SUPPLEMENT

TO

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

Volume XVII.

OFFICE OF THE REGISTRAR OF BOOKS
7-JAN 1928
MADRAS

A MEMOIR

ON

CUBIC TRANSFORMATIONS ASSOCIATED WITH A DESMIC SYSTEM

BY

R. VAIDYANATHASWAMY, M.A., D.Sc., F.R.S.E.

(Printed in co-operation with the University of Madras).

MADRAS:

PRINTED BY SRINIVASA VARADACHARI & CO.

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The Indian Mathematical Society acknowledges with thanks the financial co-operation of the Madras University in printing this Memoir.

A MEMOIR
ON THE CUBIC TRANSFORMATIONS ASSOCIATED WITH A DESMIC SYSTEM, AND THEIR APPLICATION TO PLANE GEOMETRY*

BY DR. R. VAIDYANATHASWAMY.

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* I take this opportunity to thank the Madras University for its Special Research Fellowship, during which this work was carried out.

Introduction.

The desmic configuration was noticed by Poncelet in considering the mutual relations of the centres of similitude of four spheres in space. The fact that the faces of the three tetrahedra of the configuration are extreme members of a pencil of quartic surfaces, was discovered by Stephanos, who gave the name *desmic* to the tetrahedra, as well as to the quartic surfaces of the pencil. Finally, the desmic configuration was one of the first that came in for consideration, when Reye* invented the theory of configurations with its appropriate symbolism.

The desmic configuration is closely related to the so-called Hesse configuration in the plane, consisting of three quadrangles in acyclic quadruple perspective. † This is one of the simplest of the configurations related to the plane cubic. ‡

The study of desmic surfaces was carried further in a lengthy memoir of Humbert,§ and by later writers. A brief summary of their results will be found in Jessop's *Quartic Surfaces*. The cubic complex associated with the desmic system, can also be defined as the totality of bitangents of *three* desmic surfaces. Such bitangents do not appear to have been studied so far, from the point of view of the desmic surfaces.

The cubic complex of the generating lines of a net of quadrics was investigated by Sturm, Reye, and several others. The net of quadrics with a common self-polar tetrahedron, that is to say, the net circumscribed to two desmic tetrahedra, has been studied by Meister,‖ and Reye. †† The cubic complex of generators of a net of this special type, is of great interest, as it is symmetrically related to the three tetrahedra of a desmic system, and has been investigated by Waelsch.‡‡ The related proposition,

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* Reye: *Problem der configurationen.* Acta Math, Bd. 1 ; *Über die Hexäder-Octader configuration,* ibid.

† De Vries: *Über gewisse ebene configurationen* : Acta. Math Bd. 12; also Humbert cited below

‡ See Encycl. Math Wissen ; Steinitz: *Configurationen der projectiven Geometrie*, pp. 490—492.

§ Humbert: *Sur la surface desmique du quatrieme ordre*; Louiville's Journal, 1891, Tome VII, p, 353.

‖ Meister: *Zeit. fur Math. and Phy.* 34 (1889) p. 6.

†† *Die Geometrie der Lage, Bd* III. p. 233 *ff.*

‡‡ *Sitz. Akad, Wien.* 95-II (1887) Inaccessible to the author.

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that three desmic tetrahedra are projected from any point by a cubic cone, appears to have been proved independently by Reye.*

An involutoric cubic transformation Γ_1, whose equations with respect to the tetrahedron Δ_1 of reference, are of the form

$$x'_0 : x'_1 : x'_2 : x'_3 = x_1x_2x_3 : x_0x_2x_3 : x_0x_1x_3 : x_0x_1x_2$$

has the vertices of Δ_1 for singular points; further, its eight fixed points are the vertices of two tetrahedra Δ_2, Δ_3, which form with Δ_1 a desmic system. There exist analogous cubic transformations Γ_2, Γ_3 such that Γ_2 is related in the same way to $\{\Delta_2, (\Delta_3, \Delta_1)\}$, as Γ_1 to $\{\Delta_1, (\Delta_2, \Delta_3)\}$. While transformations of the type Γ_1 have been pretty frequently used, I believe that the mutual relations of the three associated transformations Γ_1, Γ_2, Γ_3 have not been systematically studied. These form the topic of the earlier sections of this memoir; in particular, the cubic complex associated with the desmic system $(\Delta_1, \Delta_2, \Delta_3)$ appears in this connection, principally as the complex of joins of pairs of Γ_1, Γ_2 or Γ_3.

In Section (4), a method is laid down, which is very useful for dealing with quartic curves circumscribed to two desmic tetrahedra. It must be said however that this method, though not expressly stated, is yet implicit in Humbert's memoir (*l. c.*)

The next section is concerned with the intrinsic geometry on a quadric Q_3, circumscribed to Δ_1, Δ_2, in respect of the transformations Γ_1, Γ_2. This is preparatory to Section (6), which studies the (3,3) congruences Q_{31}, Q_{32}, formed by joins of pairs of Γ_1, Γ_2 situated on Q_3. These congruences are also those, which are generated by the two reguli of those quadrics of the net (Δ_1, Δ_2), whose polar reciprocals with respect to Q_3, also belong to the same net.

The general (3, 3) congruences is of two types, according as it is or is not, contained in a linear complex. The latter or the *skew* variety can be generated by ∞^1 quadric reguli; the other reguli of these ∞^1 quadrics generate a second (3,3) congruence, having the same focal surface as the first. The two congruences have the same twelve singular

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* *Loc cit.*

points and singular planes (other than the base points and base planes, if any, of the generating quadrics); the congruence lines through any singular point (in any singular plane), are composed of the lines of a plane pencil, and an isolated line not belonging to the pencil; the common focal surface of the two congruences, which is also the envelope of the generating quadrics, is of a self-dual type, of order eight and class eight, and has the same twelve singular points and planes, over and above the singularities at the base points and base planes.*

Starting from the possibility of generating two algebraically distinct congruences by the two reguli of a family of ∞^1 quadrics, Prof. Snyder† and H. Br. Owens ‡ have studied and classified the pairs of congruences generated by the reguli of rational one-parameter families of quadrics, contained in a net; they have also mentioned results for the case in which the net has eight base points. The congruences Q_{31}, Q_{32} are special types belonging to this last case; some remarkable properties of these are obtained here, such as for instance, that each of the congruences contain the edges of ∞^3 tetrahedra (with one arbitrary vertex), The treatment of these congruences is not however complete, being more or less guided by the applications to circle-geometry required in the later sections.

The sections (7) and (8) bring forward the leading idea of this memoir, which has been responsible for the previous investigations. Briefly, this idea is the interpretation of the Cartesian Plane Iscgonal Transformation in regard to a triangle $A_1A_2A_3$, as a transformation between *point-circles*, that is, as a tetracyclic point-transformation. Thus interpreted, the isogonal transformation appears as a transformation symmetrically related to the points $A_1A_2A_3$ and the point A_0, the tetra-cyclic point at infinity, carrying each of these points into the circle through the other three. Now, considering our previous three-dimensional space as the manifold of all plane circles, and the previous quadric Q_3 as the submanifold of point-circles, it turns out that the cubic trans-

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* See Ency. Math. Wiss; K. Zindler: *Algebraische Linien-Geometrie* pp. 1181, 1182, 1205, 1206.

† Snyder: *Conjugate Line-Congruences contained in a Bundle of quadric surfaces*; Trans. Am. Math, Soc., 1910.

‡ Owens: *Conjugate Line-Congruences of the Third Order*; Am. Jour. of Math, 1913, Vol. 35.

formation Γ_1 defined by the inscribed tetrahedron $\Delta_1 = A_0A_1A_2A_3$, is none other than the 'isogonal' transformation of the triangle $A_1A_2A_3$; the vertices of the associated inscribed tetrahedron Δ_2 turn out to be the four 'incentres' of the triangle $A_1A_2A_3$, and the analogous transformation Γ_2 defined in respect of Δ_2, turns out to be the plane involution of extremities of diameters of rectangular hyperbolas through the four incentres. Accordingly the properties of the congruences Q_{31}, Q_{32} admit of translation into theorems in the geometry of the triangle, the underlying identity of geometrical structure being indicated by preserving the three-dimensional nomenclature intact in the plane geometry. The class of relations between triangle and its incentres, which can be brought under this synthesis, form a well-defined group with such a marked unity of method, that it has been thought advisable to dub it with a new name, 'the tetra-cyclic geometry of the triangle.'

The sections (9) and (10) deal with several topics in the geometry of the triangle, in relation to this point of view.

The last section makes some important contributions to the apolar theory of cyclic curves—a theory, which does not appear to have advanced appreciably since Kasner's memoir of 1900 (cited below). In particular, an attempt is made to establish an analogy between the apolar and the invariant theory of the binary quartic, and that of the cyclic. For further information regarding the geometrical invariant theory of the cyclic, the following memoirs and the references quoted therein will be found useful:

E. Kasner: *The Invariant Theory of the Inversion Group* Trans. Am. Math. Soc., 1900

H. W. Turnbull: *Geometrical Interpretation of the Complete System of the (2.2) form*: Proc. Roy. Soc. Edin., 1924.

R. Vaidyanathaswamy: *On the Rank of the Double Binary Form*: Proc. Lond. Math. Soc. Ser. 2, Vol. 24.

For the rest, it must be said that, while the main idea of the memoir has been kept in view throughout, it has not been found possible for obvious reasons, to give to the geometrical topics which have arisen incidentally in connection with it, that single-minded and disinterested study

which their importance demands; this applies in particular to the (3,3) congruences, and the properties of the three types of invariant cyclic curves in relation to the triangle. It is hoped that a fuller research of these promising topics will be possible hereafter.

§ 1. Desmic Tetrahedra.

Involutoric collineations in space are of two kinds—those which have a latent point and a latent plane not passing through it,* and those which have two distinct latent lines. The former may be conveniently termed *odd involutions*, and the latter, *even involutions*. The two types may be distinguished by means of their invariant factors; the invariant factors of the former are of the shape:

$$(\lambda + \alpha),\ (\lambda - \alpha),\ (\lambda - \alpha),\ (\lambda - \alpha),$$

and those of the latter of the shape:

$$(\lambda + \alpha),\ (\lambda + \alpha),\ (\lambda - \alpha),\ (\lambda - \alpha),$$

$\alpha, -\alpha$ representing the distinct latent roots in each case.

The involutoric collineations which have four given latent points $A_0A_1A_2A_3$ are seven in number, namely the four odd involutions which have one of the points A for latent point, and the plane of the other three for latent plane, and the three even involutions each of which has a pair of opposite edges of the tetrahedron $A_0A_1A_2A_3$ for latent lines. If $A_0A_1A_2A_3$ be taken as the tetrahedron of reference for homogenous co-ordinates, $x_0x_1x_2x_3$, the equations of the four odd involutions $J_0J_1J_2J_3$ are of the form:

$$J_p :: x'_p : x'_q : x'_r : x'_s = -x_p : x_q : x_r : x_s\ (p, q, r, s = 0, 1, 2, 3),$$

and those of the even involutions $I_1\,I_2\,I_3$, of the form:

$$x'_p : x'_q : x'_r : x'_s = -x_p : -x_q : x_r : x_s\ (p, q, r, s = 0, 1, 2, 3)$$

The seven involutions J, I together with the identical collineation form an Abelian group of order 8, of which the even involutions alone form an

* For classification of collineations, reference may be made to Whitehead: *Universal Algebra*, or Hilton: *Linear Substitutions*, or Bertini: *Introduzione alla geometria degli iperspaze*. The term 'latent' is due to Whitehead.

invariant sub-group of order 4; this sub-group may be conveniently termed *the even group* of the tetrahedron $A_0A_1A_2A_3$.

The co-ordinates of an arbitrary point B_0 may be taken to be (1, 1, 1, 1) without loss of generality. Applying on B_0 the involutions of the group, we get two sets of four points $(B_0B_1B_2B_3)$ $(C_0C_1C_2C_3)$, with the co-ordinates :

$$B_0: \ (1, \ 1, \ 1, \ 1), \qquad J_0(B_0) = C_0: -(1, \ 1, \ 1, \ 1),$$

$$I_1(B_0) = B_1: \ (1, \ 1, -1, -1), \qquad J_1(B_0) = C_1: \ (1, -1, \ 1, \ 1),$$

$$I_2(B_0) = B_2: \ (1, -1, \ 1, -1), \qquad J_2(B_0) = C_2: \ (1, \ 1, -1, \ 1),$$

$$I_3(B_0) = B_3: \ (1, -1, -1, \ 1), \qquad J_3(B_0) = C_3: \ (1, \ 1, \ 1, -1)$$

Two tetrahedra related like $A_0A_1A_2A_3$, $B_0B_1B_2B_3$ are said to be *desmic* to one another; they have the property that each pair of opposite edges of either is met and separated harmonically by one pair of opposite edges of the other. The even groups of two such tetrahedra may be shewn to be permutable, and to generate a group of order 16, which when applied on an arbitrary point produces the 16 points of a Kummer configuration.*

Denoting the three tetrahedra $A_0A_1A_2A_3$, $B_0B_1B_2B_3$, $C_0C_1C_2C_3$ by Δ_1, Δ_2, Δ_3, we easily see that any two of Δ_1, Δ_2, Δ_3, are related exactly like Δ_1, Δ_2, and are desmic; the three tetrahedra $\Delta_1 \Delta_2 \Delta_3$ from a *desmic system.*

Further, it may be shewn that a pair of opposite edges of Δ_1 and that pair of opposite edges of Δ_2 which meets it, are both met the by *same* pair of opposite edges of Δ_3. It thus appears that we can construct a tetrahedron with three pairs of opposite edges chosen respectively from Δ_1, Δ_2, Δ_3. The three tetrahedra Δ'_1, Δ'_2, Δ'_3 which can be thus constructed form necessarily a second desmic system with the same 18 edges as Δ_1, Δ_2, Δ_3. The two desmic systems $(\Delta_1\Delta_2\Delta_3)$, $(\Delta'_1\Delta'_2\Delta'_3)$ are called *conjugate.* We shall also adopt the usual practice and refer to the twelve vertices of $\Delta_1\Delta_2\Delta_3$ as *the desmic points*, and to the twelve faces of $\Delta'_1\Delta'_2\Delta'_3$ as *the desmic planes.*

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* See Hudson's *Kummer's Quartic*, Chap. I.

The following known properties of the desmic system are stated here for reference :—

(1.1) Any two tetrahedra of a desmic system are in quadruple perspective, the centres and planes of perspective being the vertices and faces of the third tetrahedron of the system.

(1.2) Any face of a tetrahedron of a desmic system is the polar plane of the opposite vertex in respect of each of the two other tetrahedra of the system.

(1.3) The vertices of two desmic tetrahedra are the intersections of three quadrics which have the third tetrahedron of the system for a common self-polar tetrahedron.

The net of quadrics circumscribed to two desmic tetrahedra, contains six plane-pairs (one plane-pair through each edge of the third tetrahedron of the desmic system) ; the planes of the six plane-pairs are the twelve faces of the tetrahedra of the conjugate desmic system.

(1.4) The three tetrahedra of a desmic system and the three tetrahedra of the conjugate desmic system are all self-polar in respect of one and the same quadric.

In the co-ordinates we have been using, the equation to this quadric is

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

(1.5) The faces of three desmic tetrahedra are members of a pencil of quartic surfaces (Stephanos); therefore, the polar planes of a point in regard to three desmic tetrahedra have a common line.

The faces of the tetrahedra $\Delta'_1\ \Delta'_2\ \Delta'_3$ of the desmic system conjugate to $\Delta_1\ \Delta_2\ \Delta_3$, are represented, in the co-ordinates employed, by equations of the form :

$$(x_0^2 - x_1^2)(x_2^2 - x_3^2) = 0$$
$$(x_0^2 - x_2^2)(x_3^2 - x_1^2) = 0$$
$$(x_0^2 - x_3^2)(x_1^2 - x_1^2) = 0.$$

The pencil of quartic surfaces,— namely, *the desmic surfaces,*— of which these are members is given by:

(1.6) $$\lambda(x_0^2x_1^2+x_2^2x_3^2)+\mu(x_0^2x_2^2+x_1^2x_3^2)+\nu(x_0^2x_3^2+x_1^2x_2^2)=0$$

with $$\lambda+\mu+\nu=0.$$

The desmic surfaces (1.6) all contain the 16 lines which join the vertices of $\Delta_1\ \Delta_2\ \Delta_3$; in virtue of (1.1), these 16 lines pass by fours through the twelve desmic points, and in addition, lie by fours in the twelve desmic planes.

Hence the surfaces (1.6) have the twelve desmic points for singular points, and the twelve desmic planes for singular planes.

§ 2. The three involutoric cubic transformations defined by three desmic tetrahedra.

A cubic Cremona transformation of a general kind is obtained by making correspond to any point P, the point of intersection of the planes corresponding to P in three arbitrarily chosen correlations (Grassmann). This transformation becomes involutoric if we choose the correlations to be the polarities defined by three quadrics. By choosing the net of the three quadrics in a special manner, we obtain two types of involutoric cubic transformations which are of frequent occurrence. By choosing the net to consist of quadrics through an assigned twisted cubic, we reach one of these types; by choosing the vertices of two desmic tetrahedra as the base points of the net, (in other words, by supposing that all quadrics of the net have a common self-polar tetrahedron), we obtain the other type with which we are concerned here.

Returning to our three desmic tetrahedra $A_0\,A_1\,A_2\,A_3\,(=\Delta_1)$, $B_0\,B_1\,B_2\,B_3\,(=\Delta_2)$, $C_0\,C_1\,C_2\,C_3\,(=\Delta_3)$, let us consider the cubic transformation defined by the net of quadrics circumscribed to Δ_2, Δ_3. Taking Δ_1 as the tetrahedron of reference, and the vertices of Δ_2, Δ_3 as $(\pm1,\ \pm1,\ \pm1,\ \pm1)$, the equation to quadrics of this net is;

$$a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2=0;$$

$$a_0+a_1+a_2+a_3=0.$$

Since (x_0, x_1, x_2, x_3), $\left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right)$ are evidently conjugate points in regard to all quadrics of this form, the cubic transformation defined by the net carries (x_0, x_1, x_2, x_3) into $\left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right)$. It is clear that this transformation has the vertices of Δ_1 for singular points, each of these vertices being carried into an arbitrary point on the opposite face; the self-corresponding points of the transformation are the 8 points $(\pm 1, \pm 1, \pm 1, \pm 1)$, that is to say, they are the vertices of Δ_2, Δ_3. These data determine the cubic transformation; accordingly, we can denote the transformation by the symbol $\{\Delta_1, (\Delta_2, \Delta_3)\}$, thereby indicating that it transforms the vertices of Δ_1 into indeterminate points on the opposite faces, and the vertices of Δ_2, Δ_3 into themselves.

From this it follows that three given desmic tetrahedra Δ_1, Δ_2, Δ_3 determine *three* cubic transformations symbolised respectively by:

$$\{\Delta_1, (\Delta_2, \Delta_3)\} = \Gamma_1$$
$$\{\Delta_2, (\Delta_3, \Delta_1)\} = \Gamma_2$$
$$\{\Delta_3, (\Delta_1, \Delta_2)\} = \Gamma_3$$

The corresponding pairs of Γ_1, Γ_2, Γ_3 are, by definition, conjugate pairs in regard to the three nets of quadrics circumscribed to (Δ_2, Δ_3), (Δ_3, Δ_1), (Δ_1, Δ_2) respectively. The equations to these nets are:

$$\text{I.} \quad a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0; \quad (a_0 + a_1 + a_2 + a_3 = 0,)$$

$$(2.1) \quad \text{II.} \quad p(x_0x_1 + x_2x_3) + q(x_0x_2 + x_3x_1) + r(x_0x_3 + x_1x_2) = 0,$$

$$\text{III.} \quad p(x_0x_1 - x_2x_3) + q(x_0x_2 - x_3x_1) + r(x_0x_3 - x_1x_2) = 0.$$

Hence the transformations Γ_1, Γ_2, Γ_3 carry $x_0x_1x_2x_3$ into

$$\Gamma_1 :: \quad x_1x_2x_3 : x_0x_2x_3 : x_0x_1x_3 : x_0x_1x_2.$$

$$(2.2) \quad \Gamma_2 :: \quad x_0(x_0^2 - x_1^2 - x_2^2 - x_3^2) - 2x_1x_2x_3 : x_1(x_1^2 - x_0^2 - x_2^2 - x_3^2) - 2x_0x_2x_1 : \text{etc.}$$

$$\Gamma_3 :: \quad x_0(x_0^2 - x_1^2 - x_2^2 - x_3^2) + 2x_1x_2x_3 : x_1(x_1^2 - x_0^2 - x_2^2 - x_3^2) + 2x_0x_2x_3 : \text{etc.}$$

respectively.

Associated in a dual manner with the desmic system, we have also three cubic plane-transformations $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$ the equations to which in plane-co-ordinates are identical with (2 2.) The corresponding planes of $\overline{\Gamma}_r$ would be conjugate planes in respect of all quadrics *inscribed* to Δ_s, Δ_t $(r, s, t = 1, 2, 3)$. Since the quadric Q, in respect of which the three desmic tetrahedra are self-polar, has the form $x_0^2+x_1^2+x_2^2+x_3^2$, we may also define $\overline{\Gamma}_r$ as the transformation $Q\Gamma_rQ$, where Q represents the polarity of the quadric Q.

The transformation Γ_r is related in a simple manner to the *polar correlation* of the tetrahedron Δ_r; for instance, since the polar plane of $x = (x_0, x_1, x_2, x_3)$ in regard to the tetrahedron of reference Δ_1 has the co-ordinates $(x_1x_2x_3, x_0x_2x_3, x_0x_1x_3, x_0x_1x_2)$ it is clear that $\Gamma_1(x)$ is the pole in regard to Q of the polar plane in regard to Δ_1 of the point x. Now, from (1.5) the polar planes of x in regard to Δ_1, Δ_2, Δ_3 have a common line; therefore their poles in regard to Q are collinear. We have thus the theorem :

(2.3) *If x is any point, $\Gamma_1(x)$, $\Gamma_2(x)$, $\Gamma_3(x)$ are collinear; dually, if π be any plane, $\overline{\Gamma}_1(\pi)$, $\overline{\Gamma}_2(\pi)$, $\overline{\Gamma}_3(\pi)$ are coaxal.*

The quadrics of the three nets (2.1) will be spoken of as *the fundamental quadrics of the first, second and third systems* The system to which a fundamental quadric belongs will be generally indicated in the suffix; thus, by naming a quadric Q_r, we wish to indicate that it is circumscribed to Δ_s, Δ_t $(r, s, t = 1, 2, 3)$. In a similar manner, we shall mean by *fundamental quartic curve of the rth system*, an elliptic quartic curve circumscribed to Δ_s, Δ_t.

Invariant quadrics and quartic curves of the transformations Γ_r.

(2.4) THEOREM. *The invariant-quadrics of Γ_r are all the fundamental quadrics of the sth and t^{th} systems.* $(r, s, t = 1, 2, 3)$

For a surface of order m having multiple points of orders δ_0, δ_1, δ_2, δ_3 at the vertices of Δ_1 is transformed by Γ_1 into a surface of order $3m-\delta_0-\delta_1-\delta_2-\delta_3$. Hence a proper quadric which is transformed

into itself must be circumscribed to Δ_1. But, from the equations of Γ_1 it is clear that the circum-quadrics invariant for Γ_1 must be either of the shape:

$$p(x_0x_1 + x_2x_3) + q(x_0x_2 + x_1x_3) + r(x_0x_3 + x_1x_2),$$

or of the shape:

$$p(x_0x_1 - x_2x_3) + q(x_0x_2 - x_1x_3) + r(x_0x_3 - x_1x_2);$$

that is. they are fundamental quadrics of the second or third system.

The fundamental quartic curves of the sth system are the intersections of two fundamental quadrics of the sth system; hence they must also be invariant for Γ_r. There are however other types of quartic curves, also invariant for Γ_r. In fact, a quartic curve γ invariant for Γ_r may be either such that *all* quadrics through it, or such that only *two* quadrics through it, are invariant for Γ_r. In the former case, γ is a fundamental quartic curve of either the sth or the tth system; in the latter case, γ is not a fundamental quartic, but the intersection of a fundamental quadric of the sth system with one of the tth system. We hav thus the theorem:

(2.5) The invariant quartic curves of Γ_r comprise (1) the ∞^2 fundamental quartic curves γ_s, γ_t, (2) the ∞^4 quartic curves γ_{st} which are the intersections of quadrics of the sth system with those of the tth.

The points x, $\Gamma_r(x)$ being, by definition, conjugate points in respect of all quardrics of the rth system, it follows that the line $x\,\Gamma_r(x)$ is tangent at x to all fundamental quadrics ot the rth system which pass through x. Hence:

(2.6) The tangents at x to the three fundamental quartic curves of the three systems which pass through x, are respectively $x\,\Gamma_1(x)$, $x\,\Gamma_2(x)$, $x\,\Gamma_3(x)$.

By (2.3,) the points x, $\Gamma_1(x)$, $\Gamma_2(x)$, $\Gamma_3(x)$ are co-planar. It will be seen hereafter that the desmic surface (1.6) which passes through x contains the three fundamental quartic curves through x. Hence the plane $x\,\Gamma_1(x)\,\Gamma_2(x)\,\Gamma_3(x)$ is the tangent plane at x to the desmic surface which passes through it.

It is important to notice also that each desmic surface (1.6) is invariant for Γ_1, Γ_2 and Γ_3.

The Cubic Complex Γ.

The lines which join corresponding points of Γ_1 form a certain complex Γ. Since corresponding points p, $\Gamma_1(p)$ are by definition conjugate points in regard to all quadrics of the first system, it follows that Γ is the totality of lines cut in involution by quadrics of the first system. But a line cut in involution by quadrics of a net must be contained in one of them.* Hence Γ is also the totality of generators of quadrics of the first system. Again, the totality of generators of quadrics of a net is also the totality of chords of their curves of intersection. Hence Γ is also the totality of chords of fundamental quartic curves of the first system. Hence, if γ_1 be the fundamental quartic curve of the first system which passes through p, the complex-lines through p are the chords of γ_1 through p, and therefore form a *cubic* cone. Thus Γ is a *cubic* complex.

We see from (2.6,) that Γ may be defined more narrowly, as the totality of tangents to the fundamental quartic curves of the first system.

So far, the complex Γ has been considered only in reference to Γ_1. But as a matter of fact, Γ is symmetrically related to the desmic tetrahedra Δ_1, Δ_2, Δ_3 and is equally the complex of joins of pairs of Γ_2 or Γ_3. To prove this, it is sufficient to show that a line of Γ is also a chord of a quartic curve of the second system. Let $p\,\Gamma_1(p)$ be a line of Γ. It was shown (2.5) that every quartic curve of the second system is invariant for Γ_1. Hence the quartic curve of the second system, which

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* Let l be a line cut in involution by quadrics of the net

$$\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 ;$$

let S_r cut l in a pair of points given parametrically by the quadratic equation $f_r(t)=0$ $(r = 1, 2, 3)$. By hypothesis, the three pairs of points $f_r(t)=0$ are in involution, so that there is a linear relation of the form

$$\mu_1 f_1(t) + \mu_2 f_2(t) + \mu_3 f_3(t) \equiv 0.$$

Hence the quadric $\mu_1 S_1 + \mu_2 S_2 + \mu^3 S_3$ cuts l in a pair of points whose parameters are indeterminate, that is, contains l.

passes through p must also pass through $\Gamma_1(p)$, which proves what is required.

It follows from this that the line $p\,\Gamma_1(p)$ is cut in involution by quadrics of the second system, the pair $(p\,\Gamma_1(p)$ being one pair of the involution. Since the double points of the involution are of the form $q\,\Gamma_2(q)$, we reach the theorem which is fundamental in the theory of the cubic complex Γ:

(2.7) *Any line of the cubic complex Γ contains a pair of corresponding points of Γ_1, a pair of Γ_2 and a pair of Γ_3; and every two of these three pairs are mutually harmonic.*

From the line-co-ordinates of the join of (x_0, x_1, x_2, x_3) and $(x_1x_2x_3, x_0x_2x_3, x_0x_1x_3, x_0x_1x_2)$, we immediately find the equation of Γ to be:

(2.8) $$p_{12}p_{13}p_{23} - p_{02}p_{03}p_{23} + p_{01}p_{03}p_{13} - p_{01}p_{02}p_{12} = 0.$$

This equation shews that the twelve desmic points are singular points, and the twelve desmic planes, singular planes of Γ—a property which may be deduced directly from the definition of the complex. Further,

(2.9) the nine linear congruences, each determined by a pair of opposite edges of one of the tetrahedra Δ_1, Δ_2, Δ_3 are contained in Γ.

To prove this, we have only to observe that the equation to Γ is satisfied identically by $p_{01} = p_{23} = 0$.

There is one other important property of the cubic complex which we now proceed to obtain. One gathers from (1.3) that the six plane-pairs of each of the three nets of fundamental quadrics are different groupings of the same twelve planes, namely the twelve desmic planes:

$$(x_0^2 - x_r^2)(x_s^2 - x_t^2) = 0 \qquad (r, s, t = 1, 2, 3).$$

Denoting the faces of Δ_1', Δ_2', Δ_3' by

$$(a_0a_1a_2a_3),\ (b_0b_1b_2b_3),\ (c_0c_1c_2c_3),$$

it is easy to see that we may choose the notation in such wise that the plane-pairs

$$a_0 a_r,\ a_s a_t,\ b_0 b_r,\ b_s b_t\ \ c_0 c_r,\ c_s c_t$$

belong to the rth system of fundamental quadrics

$$(r, s\ t = 1, 2, 3).$$

Let now l be a line of Γ cutting $a_0 a_1 a_2 a_3$ in $\alpha_0 \alpha_1 \alpha_2 \alpha_3$, and let pp', qq', rr' be the pairs of Γ_1, Γ_2, Γ_3 contained in l. From our choice of notation it then follows that pp' is the common harmonic pair of $(\alpha_0 \alpha_1)$, $(\alpha_2 \alpha_3)$, qq' of $(\alpha_0 \alpha_2)$, $(\alpha_3 \alpha_1)$ and rr' of $(\alpha_0 \alpha_3)$, $(\alpha_1 \alpha_2)$. A similar result holds for the intersections of l with the faces of Δ'_2, Δ'_3. Hence

(2.10) The complex Γ may also be defined as the totality of lines which cut the faces Δ_1', Δ_2', Δ_3' (and therefore also the surfaces of the desmic pencil) in tetrads of a syzygetic involution, the squared pairs of which are the three mutually harmonic pairs of Γ_1, Γ_2, Γ_3 contained in the line.

It follows from this theorem that any complex-line is bi-tangent to three desmic surfaces, the points of contact being the pairs of Γ_1, Γ_2, Γ_3 contained in the line. Hence the cubic complex Γ is part of the bi-tangent complex of the pencil of desmic surfaces. It can be shewn that, if a line is bitangent to *two* desmic surfaces, then it is bitangent to *three*, and is a line of Γ.

It was shewn in (2.3,) that, if x be any point, the three points $\Gamma_1(x)$, $\Gamma_2(x)$, $\Gamma_3(x)$ are collinear. We can obtain further information about these three points by means of (2.10.)

Consider, in fact, the polar planes of a point x in regard to the pencil of desmic surfaces. These polar planes have a common line l; and there will be three lines of Γ which pass through x and meet l. Let xy be one of these, meeting l in y. Since xy is a complex-line, it cuts the desmic surfaces in tetrads of a syzygetic involution (2.10); also, since y is on l, the polar plane of x in respect of all the desmic surfaces passes through y. Hence the first polar triad of x in regard to *any* tetrad of the syzygetic involution must include y. But it is easy to shew

hat the only point-pairs (xy) which have this property in regard to at syzygetic involution are the three mutually harmonic squared pairs of the involution. However, it has been shewn that the squared pairs of the syzygetic involution in which a complex-line cuts the desmic pencil are the pairs of Γ_1, Γ_2 Γ_3 contained in the line. Hence it follows that the possible positions of y on l are $\Gamma_1(x)$, $\Gamma_2(x)$, $\Gamma_3(x)$. Thus these three points are collinear and the line of collinearity is the common line of the polar planes of x in regard to the desmic pencil.

The line of collinearity is also a complex-line. To shew this, we recall that Γ is the complex of chords of fundamental quartic curves. The quartic curve of the first system which passes through x is invariant for Γ_2 and Γ_3 by (2.5), and therefore passes through $\Gamma_2(x)$, $\Gamma_3(x)$. Thus $\Gamma_2(x)$ $\Gamma_3(x)$ is a chord of a fundamental quartic curve, and therefore a complex-line. Hence. :

(2.11) *The complex Γ is also the totality of the lines l which are common to the polar planes of a point in regard to the desmic pencil.*

The point x determines the corresponding line l in this theorem; but an arbitrary line l of Γ has *four* corresponding points x, namely the four residual intersections of the three fundamental quadrics, one of each system, which contain l.

Since each desmic surface is invariant for Γ_1, Γ_2, Γ_3, it follows that the desmic surface which passes through x, passes also through the three points $\Gamma_1(x)$, $\Gamma_2(x)$, $\Gamma_3(x)$. Hence these three points are part of a tetrad of the syzygetic involution in which their line cuts the surfaces of the desmic pencil. We therefore conclude :

(2.12) *The points $\Gamma_s(x)$, $\Gamma_t(x)$ separate harmonically the pair of Γ_r contained in their join.*

§ 3. The Projection—and Intersection—configuration of the Desmic system.

If p is any point, the twelve lines

$$pA_0, pA_1, pA_2, pA_3, pB_0, pB_1, pB_2, pB_3, pC_0, pC_1, pC_2, pC_3,$$

which join p to the vertices of $\Delta_1\ \Delta_2\ \Delta_3$, are generators of a cubic cone (Reye)—namely, the complex-cone whose vertex is p. Dually, since the twelve desmic planes are singular planes of Γ, the twelve lines of intersection of a plane π with the faces of Δ_1', Δ_2', Δ_3' touch a curve Σ_3 of class three—namely, the complex-curve in π.

Thus the vertices of three desmic tetrahedra project from an arbitrary point p into three plane quadrangles

$$a_0a_1a_2a_3,\quad b_0b_1b_2b_3,\quad c_0c_1c_2c_3$$

inscribed in the same cubic, which are such that any two are in quadruple perspective, the centres of perspective being the vertices of the third.* The configuration formed by such quadrangles has been studied under the name of 'Hesse configuration'; it is known that three such quadrangles must be necessarily inscribed in a cubic curve, and must be constituted by the contact points of tangents to the cubic from three collinear points on it. Thus it would follow that the tangent planes to the projecting cone along pA_0, pA_1, pA_2, pA_3 must have a common line. More precisely, we prove the theorem :

(3 1) *The tangent planes to the complex-cone with vertex p, along the lines which join p to the vertices of Δ_r meet in the line $p\Gamma_r(p)$ $(r = 1, 2, 3)$.*

For, let pA_0 cut the plane $A_1 A_2 A_3$ in p', and let A_0' be a point near A_0 in the line $A_0\,\Gamma_1(p)$. Now the line A_0p' appears in the complex-cone as a line joining A_0 and its corresponding point p' in Γ_1. But the point $\Gamma_1(A_0)$ is indeterminate, and it is only by choosing a specific

* [Two quadrangles $a_0a_1a_2a_3$. $b_0b_1b_2b_3$ can be in quadruple perspective in two ways the cyclic way, in which $a_0a_1a_2a_3$ is in perspective with

$$b_0b_1b_2b_3,\ b_1b_2b_3b_0.\ b_2b_3b_0b_1,\ b_3b_0b_1b_2.$$

and the non-cyclic way in which it is in perspective with

$$b_0b_1b_2b_3,\ b_1b_0b_3b_2,\ b_2b_3b_0b_1,\ b_3b_2b_1b_0.$$

It is the latter way which occurs here For the two types of perspective and the associated configurations, see De Vries, Uber gewisse ebene configurationen, ***Act Math.*** Bd **12**].

direction of approach to A_0 that $\Gamma_1(A_0)$ can be said to be p'. This direction of approach is $A_0' A_0$. Now there is a pair of corresponding points of Γ_1 on each generator of the complex-cone; it is therefore clear that the locus of such points must pass through A_0 in the direction A_0A_0'. Hence the tangent plane to the complex-cone along pA_0 contains $\Gamma_1(p)$, which proves the theorem.

The lines through p meeting pairs of opposite edges of Δ_r are also complex-lines (2.9). In the projection, these lines will give the harmonic triangle of the corresponding quadrangle. This leads from a different point of view, to the property of the projected configuration. For, it is seen hereby that the desmic tetrahedra Δ_1, Δ_2, Δ_3, are projected into three quadrangles, which, *together with their harmonic triangles*, are inscribed in the same cubic curve. It is easy to shew that the only inscribed quadrangles of a plane cubic which enjoy this property, are those constituted by the contact points of four tangents from a point on the curve.

From the property of conjugate desmic systems, it follows that $\Delta_1'\Delta_2'\Delta_3'$ project from p into three quadrangles which have the same nine pairs of opposite sides as the three quadrangles into which $\Delta_1\Delta_2\Delta_3$ project.* Hence the harmonic triangles of the former three quadrangles are identical in the ensemble of vertices, with the harmonic triangles of the latter.

It can also be shewn that the three quadrangles obtained by adjoining the projection of $\Gamma_r(p)$ to the harmonic triangle of the quadrangle which is the projection Δ_r, also form a quadruple-perspective configuration of the same type.†

Results dual to these hold for the configuration in which a plane π intersects the desmic system.

We shall now study the intersection-configuration on the plane π in reference to the three nets of conics $N_1N_2N_3$ in which π cuts the

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* Cf. De Vries. *loc. cit.*

† See Gabbat: On the generalisation of the Theory of circles associated with a triangle by means of the theory of plane cubic curves, *Proc. Camb. Phil. Soc.*, Vol. XXI, 1922.

three systems of fundamental quadrics. Let Σ represent the envelope of complex-lines in π; then, as already remarked Σ is a curve of class three inscribed in the three quadrilaterals $\delta'_1, \delta'_2, \delta'_3$ in which π cuts the faces of $\Delta'_1, \Delta'_2, \Delta'_3$ Let us further denote by δ_r the quadrilateral in which π intersects Δ_r, and by h_r, h'_r the harmonic triangles of δ_r, δ'_r $(r = 1, 2, 3)$. It has been shewn above that the quadrilaterals δ_r have the same nine pairs of opposite vertices, and consequently the nine sides of the triangles h_r are the same as those of h'_r.

A net N of conic-loci possesses a co-variant cubic S, which is the locus of vertices of line-pairs of N, and also the locus of point-pairs pq which are conjugate points in regard to all conics of N. It also possesses a contravariant cubic envelope Σ, which may be defined as the envelope of line-pairs of N, or as the envelope of lines cut in involution by the conics of N, or as the envelope of lines pq which are the joins of conjugate pairs of N. A cubic S is in general the co-variant cubic of three nets of conics; any cubic envelope Σ is also the contravariant cubic of three nets.*

Any complex-line in the plane π contains a pair of corresponding points of each of the transformations $\Gamma_1, \Gamma_2, \Gamma_3$ (2.7), and is cut in involution by each of the nets of fundamental quadrics, and therefore by each of the conic-nets N_1, N_2, N_3. Hence:

(3.2) *The three nets of conics, N_1, N_2, N_3 have the same contravariant cubic Σ, namely, the complex-curve in π. Any tangent to Σ contains a pair of conjugate points of each of the nets $N_1 N_2 N_3$; and the three such pairs are mutually harmonic.*

From the harmonic property it follows that all conics of N_r which pass through a point p, pass also through three other points which are situated on the three tangents to Σ through p, and are the harmonic conjugates of p in respect of the pairs of Γ_r contained in the tangents.

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* For an account of the theory of nets of conics and the associated cubics, see *Ency. Math. Wiss.* III. C. 1. and III. C. 5.

Let l be the tangent to Σ at the point λ, and let $l_r\, l'_r$ be the pair of Γ_r contained in l $(r = 1, 2, 3)$. Further, let the line which together with l makes up a line-pair of N_r cut l in l''_r $(r = 1, 2, 3)$. We have the important result:

(3.3) *The four points λ, l_r'' $(r = 1, 2, 3)$ form a tetrad of the syzygetic involution determined by the three mutually harmonic pairs $l_r l_r'$.*

To prove this, consider the tangent to Σ which is consecutive to l; it is clear that this tangent cuts l and the line which with l makes up a line-pair of N_r in two points, which tend in the limit to the positions λ and l_r''. But the intersections of any tangent L to Σ with a conic of N_r will be harmonically conjugate with respect to the pair of Γ_r in L. Thus $\lambda\ l_r''$ is harmonic to $l_r\, l_r'$, which establishes the theorem. It follows at the same time that $l_r''\, l_s''$ is harmonic to $l_t\, l_t'$ $(r, s, t = 1, 2, 3)$.

Let now $p_s q_t$, $p_s q_t'$ be a line-pair of N_s, and let p_s' be the point conjugate to p_s in regard to N^s. The conics of N_r which pass through p_s, pass also through three other fixed points, of which one is obviously p_s'. The two other points should lie on $p_s q_t$, $p_s q_t'$; let them be q_t, q'_t, so that $p_s q_t$ separate harmonically the pair of Γ_r which lies in their join. It then follows from (3.3) that *q_t is the vertex of a line-pair of N_t, one line of the line pair being $p_s q_t$.* Since q_t, q_t' are thus vertices of line-pairs of N_t and are further base-points of a pencil of conics of N_r, it follows that *q_t, q_t' are conjugate points with respect to N_t.* It follows by parity of reasoning that $(p_s q_t, p'_s q_t)$, $(p_s q_t', p_s' q_t')$ are the line-pairs of N_t with the vertices q_t, q_t', and also that $(p_s' q_t, p_s' q_t')$ is a line-pair of N_s. It also results that $(p_s q_t', p_s' q_t')$, $(p_s q_t', p_s' q_t)$ are line-pairs of N_r.

From this result we can derive a remarkable property of the **nets** N_1, N_2, N_3, namely

(3.4) *Any conic n_r of N_r contains three pairs of Γ_s and three pairs of Γ_t; these pairs are so related that each of the pairs of Γ_s is harmonic to two of the pairs of Γ_t.*

For, since the locus of points p_s which are vertices of line-pairs of N_s is a cubic, the conic n_r contains six such points. But, whenever n_r passes through p_s, it also passes its conjugate point p_s' in respect of N_s. Thus n_r contains three conjugate pairs P_sP_s', Q_sQ_s', R_sR_s' (say) of N_s, that is to say, three pairs of Γ_s, and three conjugate pairs P_tP_t', Q_tQ_t', R_tR_t' of N_t. The notation may be chosen so that

$$(P_sP_t, P_sP_t'), (Q_sQ_t, Q_sQ_t'), (R_sR_t, R_sR_t')$$

are line-pairs of N_s. Now, all conics of N_s which pass through P_t must also pass through P_t' and form a pencil; hence their two other intersections with n_r constitute pairs of an involution. Further, the conic of N_s which passes through P_t, Q_t must also pass through the conjugate points P_t', Q_t'. Hence Q_tQ_t', and similarly R_tR_t' are members of the involution. But, from the fact that (P_sP_t, P_sP_t'), $(P_s'P_t, P_s'P_t')$ are line-pairs of N_s, it follows that the double points of the involution are P_sP_s'. Thus P_sP_s' is harmonic to Q_tQ_t', R_tR_t', which proves our theorem. The result also shews that the two triangles

$$(P_sP'_s, Q_sQ'_s, R_sR'_s), (P_tP'_t, Q_tQ'_t, R_tR'_t)$$

are polar triangles of one another with respect to n_r.

The tangents to the class-cubic determine a symmetric (3.3) correspondence between the points of any conic n. Our result shews that, whenever n belongs to one of the nets N_1, N_2 N_3, the (3,3) correspondence determined on it admits a closed tetrad—namely a quadrangle of the type $(P_sP'_s\ P_tP'_t)$ all of whose six sides belong to Σ. The existence of such a tetrad is a poristic property of the correspondence;* hence a conic of any of the nets $N_1N_2N_3$ admits ∞^1 inscribed quadrangles all of whose sides touch the involution-envelope Σ. These quadrangles are obviously the intersections of the conic with other conics of the same net, and form a general binary involution of the fourth order. We can

* For an algebraic proof of this, see A. B Coble: Symmetric binary forms and multiple involutions, *Am. Jour. of Math.*, 47.

accordingly infer from the above a curious theorem concerning pencils of binary quartics:

(3.5) *Any pencil $\lambda_1 F_1(x) + \lambda_2 F_2(x)$ of binary quartics contains three members of the form $f_r(x) f'_r(x)$ $(r = 1, 2, 3)$, where f_r, f'_r are quadratics such that f_r, f'_s are apolar for $r \neq s$.*

This theorem is evidently the extension to the general pencil of binary quartics of the property enjoyed by the syzygetic pencil, of possessing three perfect squares.

§ 4. The Geometry of the Fundamental Quartic Curves.

Elliptic Representation of a Fundamental Quartic.

Consider a fundamental quartic curve γ_3 of the third system—that is, one that is circumscribed to $\Delta_1 \Delta_2$. It is well-known that the co-ordinates of points on a quartic curve of genus *one* can be exhibited as elliptic functions of a parameter u, in such wise that the condition for co-planarity of four points whose parameters are $u_1 u_2 u_3 u_4$ is of the form.*

$$u_1 + u_2 + u_3 + u_4 = 0 \quad \text{(mod. periods)}.$$

Using this representation, let u, u', u'', u''', v, v', v'', v''' be the parameters of the vertices of Δ_1, Δ_2 on γ_3. The desmic relationship of Δ_1, Δ_2 yields immediately the following equations between these parameters:

$$\left.\begin{aligned} u + u' + v + v' &= 0 \\ u + u' + v'' + v''' &= 0 \\ u'' + u''' + v + v' &\equiv 0 \\ u'' + u''' + v'' + v''' &= 0 \end{aligned}\right\} \text{(mod. periods)}.$$

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* See *Ency. Math. Wiss.* III 2_2.

so that, we have

$$\left.\begin{aligned} u + u' &= u'' + u''' = -(v + v') = -(v'' + v''') \\ u + u'' &= u' + u''' = -(v + v'') = -(v' + v''') \\ u + u''' &= u' + u'' = -(v + v''') = -(v' + v'') \end{aligned}\right\} \text{(mod. periods).}$$

This shews that

$$u = u' - u = u'' - u' = u''' - u - u'$$

$$-u = v = v' - u = v'' - u' = v''' - u - u'$$

From these we have $2u + 2v = 0$, which shews that the tangents to the fundamental curve γ_3 at each of the vertices of Δ_1 meets the tangents at each of the vertices of Δ_2. Hence:

(4.1) *The tangents to γ_3 at the vertices of Δ_1, Δ_2 are generators of opposite systems of a certain fundamental quadric containing the curve.*

More generally it is easy to shew that the chords of the curve whose extremities u, v are connected by the relation $u + v$ = constant $= k$ constitute a quadric regulus, the conjugate regulus being given by $u + v = -k$. Corresponding points (U, U_1), (U, U_2) of Γ_1, Γ_2 on the curve are given by the respective relations

$$U + U_1 = -2u$$

$$U + U_2 = \quad 2u:$$

for, the pairs of Γ'_1 form a birational involution on the curve with four self-corresponding points, namely v, v' v'', v'''. From a known theorem on the types of such involutions on curves of genus one, it follows that $U + U_1$ = constant. Taking $U = v = -u$, we have $U_1 = U$, so that $U + U_1 = -2u$. This result shews that:

(4.2) *The joins of pairs of Γ_1, Γ_2, on the curve are generators of different systems of the same quadric—namely the fundamental quadrics of (4,1).*

Point-representaion of fundamental quartic curves γ_3.

Some of the properties pertaining to the desmic pencil, and the net of fundamental quadrics are best stated with reference to the fundamental quartic curve of a particular system as the space-element.

The fundamental quadrics of, say, the third system, can be represented without exception or repetition by the ratios of three homogenous parameters, and therefore constitute a ternary domain. The fundamental quartic curves of the third system may be considered to be representative of *pencils* of fundamental quadrics, and therefore constitute the contragredient ternary domain.

For simplicity of conception and expression, we may consequently represent the quartic curves of the third system by points in a plane π; the quadrics of the third system will then be represented by the lines of π. The incidence-relation of line and point in π will then correspond to the incidence-relation of fundamental quadric and fundamental quartic curve.

There are four special quartic curves of the third system, namely the four tetrads of lines concurrent in a vertex of Δ_3; these will correspond to four special points $\lambda_0\lambda_1\lambda_2\lambda_3$ in the plane π. Each of the twelve desmic planes passes through an edge of Δ_3 and contains four of the sixteen lines—namely, two through each vertex in the edge. Each of the six plane-pairs in the net of fundamental quadrics is composed of two desmic planes meeting in an edge of Δ_3, and is easily seen to contain the two tetrads of concurrent lines through the two vertices in the edge. Hence *the six plane-pairs correspond to the six sides of the quadrangle* $\lambda_0\lambda_1\lambda_2\lambda_3$. Each of the vertices of the harmonic triangle $\mu_1\mu_2\mu_3$ of the quadrangle $\lambda_0\lambda_1\lambda_2\lambda_3$ will then correspond to the three degenerate quartic curves, each composed of a pair of opposite edges of Δ_1 and the pair of opposite edges of Δ_2 which meets it.

Humbert shews* that the desmic surfaces (1 6) cut a fundamental quadric of any system in a pair of quartic curves. This is clearly equivalent to the theorem :

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* *loc. cit*

(4.3) *The desmic pencil of quartic surfaces (1.6) corresponds in our representation to the pencil of conics through* $\lambda_0\lambda_1\lambda_2\lambda_3$. *The three desmic surfaces which reduce to the faces of* Δ'_1, Δ'_2, Δ'_3 *correspond to the line-pairs* $(\lambda_0\lambda_1,\ \lambda_2\lambda_3)$, $(\lambda_0\lambda_2,\ \lambda_3\lambda_1)$, $(\lambda_0\lambda_3,\ \lambda_1\lambda_2)$.

The net of fundamental quadrics contains four pencil of cones, which are represented by lines through one of the points $\lambda_0\lambda_1\lambda_2\lambda_3$. Now the *modulus* of a quartic curve is the absolute invariant of the binary quartic which gives the parameters of the four quadric cones containing the curve. Hence if γ is the point in π which corresponds to the quartic curve γ_3, then the modulus of γ_3 is the absolute invariant of the pencil $(\gamma, \lambda_0\lambda_1\lambda_2\lambda_3)$. Hence if γ_3 varies continuously so as to have a constant modulus, locus of γ is a conic through $\lambda_0\lambda_1\lambda_2\lambda_3$, so that from (4.3), we have :

(4.4) *A desmic surface is the locus of fundamental quartic curves of constant modulus.*

A *side* of $\lambda_0\lambda_1\lambda_2\lambda_3$ corresponds, as already remarked, to a pair of planes. A point on such a side, being the intersection of the side and some other line, will correspond to a pair of conics.

It is clear that we may translate the polar theory of conics in the plane π into a polar theory of fundamental quartic curves in respect of the desmic surfaces Thus through an assigned fundamental quartic curve γ_3 we can pass two quadrics each of which touches a given desmic surface D along a fundamental quartic curve; the quadric containing the two curves of contact is the 'polar' quadric of γ_3 with respect to D. If γ'_3 is a fundamental curve on the polar quadric of γ_3, then γ_3, γ_3' are 'conjugate' in regard to D. The quadric through γ_3, γ'_3 is cut by D in two quartics harmonically separating γ_3, γ'_3, etc. Also, any fundamental quadric Q_3 is cut by the desmic surfaces in pairs of quartic curves in involution, and is therefore touched along two quartic curves ε_3, ε'_3 (say), by two of the desmic surfaces. The two curves ε_3, ε'_3 have the property of being conjugate curves in respect of every desmic surface. The properties of pairs of curves like ε_3, ε'_3 (which we shall

call 'conjugate quartic curves') are of great importance, and form our next topic.

Conjugate Quartic Curves.

The structure of the ternary domain of fundamental quartic curves is dependent on the ternary relations in the plane π established by the quadrangle $\lambda_0\lambda_1\lambda_2\lambda_3$; the most important among these is the Cremona quadratic involution with the self-corresponding points $\lambda_0\lambda_1\lambda_2\lambda_3$ and singular points at the vertices $\mu_1\mu_2\mu_3$ of the harmonic triangle. The pairs of this involution are conjugate points in respect of all conics through $\lambda_0\lambda_1\lambda_2\lambda_3$ and correspond to what we have termed 'conjugate fundamental quartic curves.' We proceed to obtain the geometrical relationship between such pairs of quartic curves.

The system of fundamental quartic curves γ_3 is in homographic correspondence with the system of their tangents at any one of their common points ($A_0\,A_1\,A_2\,A_3\,B_0\,B_1\,B_2\,B_3$)—say, at A_r. In this correspondence, the four special quartic curves which reduce to a tetrad of lines concurrent at a vertex C_s of Δ_3, correspond to the lines $A_r\,C_s$ $(s = 0, 1, 2, 3)$. Also, the three curves which consist of two edges of Δ_1 and two, of Δ_2, correspond to the three lines $A_r\,A_p$, $A_r\,A_q$, $A_r\,A_s$ $(p,q,r,s$ being all different). The former four quartic curves are those which correspond to $\lambda_0\,\lambda_1\,\lambda_2\,\lambda_3$ in π, and the latter three, to $\mu_1\,\mu_2\,\mu_3$ the the vertices of the harmonic triangle of $\lambda_0\,\lambda_1\,\lambda_2\,\lambda_3$.

Now, there is a Cremona quadratic involution I among the lines through A_r, namely, the involution of pairs of generators through A_r to a fundamental quadric of the third system. When a quadric Q_3 contains the line $A_r\,C_s$, it clearly becomes a cone with vertex at C_s, and therefore its other generator through A_r coincides with $A_r\,C_s$ $(s=0, 1, 2, 3)$. Hence the four lines $A_r\,C_s$ $(s = 0, 1, 2, 3)$ are self-corresponding lines of I. Also, since it is only one condition on the quadric Q_3 to contain the line A_rA_p, it follows there the are ∞^1 fundamental quadrics containing $A_r\,A_p$, their equations being:

$$\lambda\,(X_r\,X_q + X_p\,X_s) + \mu\,(X_r\,X_s + X_pX_q) = 0.$$

It is easily seen that the locus of the other generator through A_r of

these quadrics is the plane $A_r A_q A_s$. Thus the three lines $A_r A_p$, $A_r A_q$, $A_r A_s$ are singular lines of I, each of them corresponding in I to an indeterminate line of the pencil determined by the other two.

The Cremona involution I leads, through the homography between the lines through A_r and the fundamental curves touching them, to an involution I′, between the fundamental curves. From what has been proved concerning I, it follows that I′ is a quadratic involution with the curves corresponding to $\lambda_0\, \lambda_1\, \lambda_2\, \lambda_3$ for self-corresponding elements, and those corresponding to $\mu_1 \mu_2 \mu_3$ for singular elements. Hence I′ is equivalent to the involution of conjugate pairs determined by the quadrangle $\lambda_0\, \lambda_1\, \lambda_2\, \lambda_3$ — that is to say, it is the involution of conjugate quartic curves. And this result holds whichever of the eight common points is represented by A_r. Thus, conjugate curves have the property that their two tangents at any common point are generators of the *same* fundamental quadric. Since the eight tangents at the base points of a fundamental curve are, by (4.1), generators of a unique quadric, we obtain the theorem :

(4.5) Conjugate fundamental quartic curves have the property that their sixteen tangents at the base points are all generators of the fundamental quadric containing them.

Hence, from (4.2), the fundamental quadric which contains two conjugate curves γ_3, γ_3', is the locus of lines which join pairs of Γ_1 or Γ_2 on γ_3, γ'_3. It is untrue in particular cases, and therefore in general, that a generator of the quadric cuts both γ_3 and γ'_3 in pairs of the same transformation Γ_1 or Γ_2. Hence every generator of the quadric cuts *one* of the conjugate curves γ_3, γ'_3 in a pair of Γ_1 and the *other* in a pair of Γ_2. Hence, by (2.7),

(4.6) Two conjugate fundamental curves γ_3 γ_3' are cut by every one of their common chords in pairs of Γ_1, Γ_2, and therefore harmonically.

This theorem includes (4.5) as a limiting case.

Generally, an elliptic quartic curve γ_3 on a quadric Q_3 is touched by four generators of each system—say, at the points $A_0\, A_1\, A_2\, A_3$,

$B_0 B_1 B_2 B_3$. The generators through these eight points other than the tangents to γ_3 are tangents to a second quartic curve γ'_3, the two curves γ_3, γ'_3 having the property of cutting every generator harmonically. The tetrahedra $A_0A_1A_2A_3$, $B_0B_1B_2B_3$ are necessarily desmic, and the curves γ_3, γ_3' are conjugate fundamental curves in respect of them.

The curve γ_3 establishes a (2,2) correspondence between the parameters x, y of the generators of the two systems which meet upon it; accordingly, it has a symbolic double-binary equation of the form :

$$a_x^2 b_y^2 = 0.$$

It is well-known that the equation of the conjugate curve γ'_3 is :

$$c_x^2 d_y^2 \equiv (a_x^2 b_y^2, \ a_x^2 b_y^2)^{1,1} = 0.$$

The reciprocal relation between conjugate curves shews the reciprocity between a (2,2) form and its (1,1) transvectant—a well-known result in double-binary algebra.*

The Rational cubic series of quadrics K defined by a fundamental quadric Q_3.

Take any line Q_3 in π, and consider the envelope K of lines which join the points of Q_3 to their conjugates in the Cremona involution determined by the quadrangle $\lambda_0\lambda_1\lambda_2\lambda_3$. It is easily seen that K is a rational envelope of the third class, and has Q_3 for a bitangent at the unique pair of conjugate points contained in Q_3 and also touches the six sides of the quadrangle $\lambda_0\lambda_1\lambda_2\lambda_3$.

The series K of fundamental quadrics which corresponds to the envelope K is therefore a rational cubic series which is completely determined by Q_3 and has Q_3 for its double quadric. From the definition of the envelope, it is seen that the series K consists of all quadrics of the net, one of the two fundamental conjugate curves on which is contained in Q_3. Further, the cubic series also comprises the six plane-pairs of the net. We shall now shew that the series alse consists of

* See Kasner, Turnbull and Vaidyanathaswamy, quoted in the Introduction.

those quadrics of the net whose polar reciprocals with respect to Q_3 also belong to the net.

Let γ be any fundamental quartic curve on Q_3, γ' its conjugate curve. Then by definition, the quadric (γ, γ') is a quadric K of the cubic series. Since γ, γ' are conjugate curves on K, the tangent at A_0 to γ is a generating line of K, But, since γ is on Q_3, this tangent lies in the tangent plane at A_0 to Q_3. Hence this tangent plane (and in the same manner, the tangent plane to Q_3 at any vertex of Δ_1 or Δ_2) contains a generator of K, and so touches K. Thus the cubic series K consists of quadrics which pass through the vertices of Δ_1 Δ_2, and touch the tangent planes at these vertices to Q_3; that is to say, K is the one-parameter family in the net, which is reciprocated into itself by Q_3. This proves the statement made above. It also follows that the family of quadrics K is a cubic family, not only in point-co-ordinates, but also in plane-co-ordinates, in other words, there are three quadrics K which pass through an arbitrary point, and three, which touch an arbitrary plane. The three quadrics K which pass through a point p have a common curve of intersection—namely, the fundamental quartic curve through p; dually, the three quadrics K which touch a plane are inscribed in the same quartic developable.

From (4.1), the two sets of four tangents at the vertices of Δ_1 Δ_2, to a fundamental quartic curve on Q_3 are generators of opposite systems of a fundamental quadric, which, from what has preceded is a quadric of the cubic series K. It follows therefore that we can associate in a continuous manner the two reguli of a variable quadric K of the series with the two tetrahedra Δ_1, Δ_2; this association shews also the necessity of the fact already proved, that Q_3 is the double quadric of the cubic series. From this we infer that the two reguli of the quadrics K form two algebraically distinct congruences Q_{31}, Q_{32}.* Since K is a cubic series both in point and in plane co-ordinates, each of the congruences Q_{31}, Q_{32} must be of order-class (3,3); since K is its own polar reciprocal series with respect to Q_3, each of the congruences

* See Snyder and Owens, *loc cit.*

Q_{31}, Q_{32} is its own polar congruence in regard to Q_3. These congruences are also bicursal, that is, the co-ordinates of the general line of Q_{3r} can be put as rational functions of two parameters λ, μ, of which one fixes the quadric K, and the other, the position of the line on the quadric so fixed.

The envelope K in the plane π, is, *qua* locus, a quartic. This must correspond to an octavic surface F_8 which is the envelope of the quadrics of the cubic series K, and therefore the focal surface of each of the congruences Q_{31}, Q_{32}. It is clear that F_8 has quadruple points at the vertices of Δ_1, Δ_2.

We study hereafter the congruences Q_{31}, Q_{32} under the name of 'fundamental congruences', and shall then have occasion to return to the cubic series.

Singular points and planes of the congruences Q_{31}, Q_{32}.

We have defined the two congruences Q_{31}, Q_{32}, as those generated by the two reguli of quadrics of the cubic series K; the envelope F_8 of this series is accordingly the common focal surfaces of the two congruences. Since K is, by definition, its own polar reciprocal family in regard to Q_3, it follows that F_8 is its own polar reciprocal in regard to Q_3. Hence F_8, is also of class eight.

The eight vertices of Δ_1, Δ_2 are base points of the family K. The K-generators which pass through one of these, for instance, A_0, either lie in the tangent-plane at A_0, or in the cone which projects the section of Q_3 by the plane $A_1A_2A_3$, according as they belong to one or the other regulus; the former generators belong to Q_{32}, the latter to Q_{31}. Thus the vertices of Δ_1 (Δ_2) are singular points of the second order of Q_{31} (Q_{32}), while the vertices of Δ_2 (Δ_1) are singular points of the first order.

Since the envelope K in the representative plane touches the six sides of the quadrangle $\lambda_0\lambda_1\lambda_2\lambda_3$, it follows that the cubic series contains the six plane pairs of the net. The reguli of a quadric which is a plane-pair, must consist of two plane pencils; in the present case how-

ever, each plane-pair *qua* member of the family K, may be considered as a pair of plane pencils in two distinct ways. To shew this, consider the two pairs of desmic planes (D_{01}, D'_{01}), (D_{23}, D'_{23}), which pass through two opposite edges C_0C_1, C_2C_3 of Δ_3. Thus, we can take:

D_{01} for the plane $C_0\,C_1\,A_0\,A_1\,B_0\,B_1$,
D'_{01} ... ... $C_0\,C_1\,A_2\,A_3\,B_2\,B_3$,
D_{23} ... ... $C_2\,C_3\,A_0\,A_1\,B_2\,B_3$,
D'_{23} ... ... $C_2\,C_3\,A_2\,A_3\,B_0\,B_1$.

Let d_{01}, d'_{01}, d_{23}, d'_{23} be the poles in regard to Q_3 of D_{01}, D_{01}', D_{23}, D'_{23}, so d_{01}, d'_{01} lie on $C_2\,C_3$ and d_{23}, d'_{23} on $C_0\,C_1$. Now, if nothing is known concerning the quadric constituted by the two planes $D_{01}\,D'_{01}$, then we can consider its two reguli to be any two plane pencils with vertices on $C_0\,C_1$, situated respectively in D_{01}, D'_{01}. Since however the quadric is to belong to the K-series, the choice of the plane pencils is restricted by the condition, that the pair of polar reciprocal plane pencils should also constitute a K-quadric, which can only be the pair of planes D_{23}, D'_{23}. Hence there are only four possible choices of the plane pencils, for the reguli of (D_{01}, D'_{01}); namely, either the pair of pencils (d_{23}, D_{01}), (d'_{23}, D'_{01}), or the pair (d'_{23}, D_{01}), (d_{23}, D'_{01}), or the pair (d_{23}, D_{01}), (d_{23}, D'_{01}), or lastly the pair (d'_{23}, D_{01}), (d'_{23}, D'_{01}). We may easily find to which of the two congruences these plane pencils belong. Thus since d_{23} is the pole of D_{23}, the line $d_{23}\,A_0$ is a tangent at A_0; thus the pencil (d_{23}, D_{01}) should belong to Q_{32}. Similarly, (d_{23}, D'_{01}) belongs to Q_{31}, (d'_{23}, D_{01}) belongs to Q_{31}, and lastly (d'_{23}, D'_{01}) belongs to Q_{32}. Thus, in the first two choices, the two reguli of the degenerate quadric belong to the *same* congruence. Hence only the last two choices are admissible. Hence it follows that each of the two congruences has the twelve desmic planes for singular planes, and the twelve poles of these planes in regard to Q_3, for singular points. Further any desmic plane D_{01} contains an isolated line A_0A_1 of Q_{31}, not belonging to the plane pencil (d'_{23}, D_{01}). Similarly, the lines of Q_{31} through a singular point d_{23} comprise the lines of the plane pencil (d_{23}, D'_{01}), and an isolated line, namely, the polar line of A_0A_1. Similar remarks hold for

Q_{32}. This verifies a general theorem on the skew type of (3.3) congruence.*

The focal surface possesses the twelve singular points for conical points, and the twelve desmic planes for singular planes. The intersection of the focal surface with a desmic plane D_{01}, consists of three conics through $A_0A_1B_0B_1$, one of which is doubly counted; the doubly counted conic passes through d_{23}, d'_{23}.

The equation to the focal surface F_8 may be found by taking Δ_1 as the tetrahedron of reference, and the equation of Q_3 in the form :

$$Q_3 : p\,(q^2 - r^2)\,(X_0X_1 - X_2X_3) + q\,(r^2 - p^2)\,(X_0X_2 - X_3X_1) + r\,(p^2 - q^2)\,X_0X_3 - X_1X_2).$$

It is then easily shewn that the quadrics of the cubic series K are of the form

$$\lambda\,(X_0X_1 - X_2X_3) + \mu\,(X_0X_2 - X_3X_1) + \nu\,(X_0X_3 - X_1X_2), \text{ where}$$

$$\lambda = (qr + pt)\,(t^2 - p^2)\,(q^2 - r^2)$$
$$\mu = (rp + qt)\,(t^2 - q^2)\,(r^2 - p^2)$$
$$\nu = (pq + rt)\,(t^2 - r^2)\,(p^2 - q^2),$$

where t is a parameter. Writing $\xi_r = X_0X_r - X_sX_t$ $(r, s, t = 1, 2, 3)$, the equation of the focal surface is.

$$\Delta_t\,[\lambda\,\xi_1 + \mu\,\xi_2 + \nu\,\xi_3] = 0,$$

where Δ_t denotes the t — discriminant.

§ 5. Geometry of the Transformations Γ_r, Γ_s on a common invariant quadric Q_t.

All quadrics Q_3 have been shewn to be invariant for Γ_1 and Γ_2. We shall choose any particular fundamental quadric Q_3 and study the behaviour of the transformations Γ_1, Γ_2 upon it.

Let us call, for convenience, the two systems of generating lines of Q_3, *the* λ*-system* and the μ*-system*. A λ-generator l is transformed by

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*See Introduction.

Γ_1 into a twisted cubic L through $A_0A_1A_2A_3$, lying on the quadric. L would belong to the system of twisted cubics which cut the λ-generators twice, the intersections of l and L constitute the pair of Γ_1, which l *qua* complex-line should contain. Thus Γ_1 establishes a homography between the λ-generators l and the twisted cubics L; this homography may be determined from the four particular cases in which l passes through a vertex of Δ_1. If, for instance, l passes through A_0, it is clear that L splits into the μ-generator through A_0 and the conic in which the plane $A_1A_2A_3$ cuts Q_3.

If now, l is the λ-generator through a vertex B_r of Δ_2, then the pair of Γ_1, which l *qua* complex-line contains, is the pair B_r B_r. Hence the corresponding twisted cubic L passes through $A_0A_1A_2A_3B_r$ and touches the λ-generator l at B_r. Hence :

(5.1) *The points A_r, B_r are so related with respect to Q_3, that each of the two twisted cubics on Q_3 through $A_0A_1A_2A_3B_r$ touches one of the two generators through B_r (r = 0, 1, 2, 3).*

Another relation between the points A_r, B_r, has been indicated in the previous section. It may be stated as follows :—

(5.2) *The two elliptic quartic curves on Q_3, which touch the generators of one system at the points $A_0A_1A_2A_3$ necessarily touch the generators of the other system at the points $B_0B_1B_2B_3$.*

We have shewn in (4.6) that the common chords of two conjugate quartic curves cut them in pairs of Γ_1, Γ_2 and therefore harmonically. From (4.2) we easily infer the theorem :

(5.3) *The locus of a point p on Q_3, such $p\,\Gamma_1(p)$ is a λ-generator, is also the locus of points q such that $q\,\Gamma_2(q)$ is a μ-generator, and is a fundamental quartic curve γ; similarly, the locus of points r such that $r\,\Gamma_1(r)$ is a μ-generator is also the locus of points s such that $s\,\Gamma_2(s)$ is a λ-generator, and is a fundamental curve γ'; γ and γ' are the unique pair of conjugate fundamental curves contained in Q_3.*

Consider now a λ-generator l, which cuts γ in $p\ \Gamma_1(p)$ and γ' in $q\ \Gamma_2(q)$. It is clear that pairs of points $s\,t$ on l, such that $\Gamma_2(s)$ lies on the μ-generator through t, must be homographically related;* say, $t = \mathrm{H}(s)$. Now, since $q\ \Gamma_2(q)$ is a λ-generator, (5.3) shews that $q\ \Gamma_1(q)$ must be a μ-generator. Thus q, $\Gamma_2(q)$ are the double points of the homography H. Since H obviously interchanges $p\ \Gamma_1(p)$, it follows that H is the involution of point-pairs harmonic to $q\ \Gamma_2(q)$. Hence:

(5.4) If l is a λ-generator cutting γ' in $q\ \Gamma_2(q)$, any point-pair st harmonic to $q\ \Gamma_2(q)$ has the property that $\Gamma_1(s)$ lies on the μ-generator through t, and $\Gamma_1(t)$ on the μ-generator through s

Double-binary equations to Γ_1, Γ_2 on Q_3.

From the preceding may be derived the double-binary equations of Γ_1, Γ_2. Let p be any point on Q_3, and p_1, p'_1, p_2, p_2' the harmonic conjugates of p in regard to the respective point-pairs in which the λ- and the μ-generator through p cuts γ, γ'. Then by (5.4), $\Gamma_1(p)$ lies on the μ-generator through p_1', and also on the λ-generator through p_2'; similarly $\Gamma_2(p)$ lies on the μ-generator through p_1, and on the λ-generator through p_2'. If the λ- and μ-generators are represented by two digredient binary parameters x and y, then the equations of the conjugate curves γ, γ' are of the form:

$$a_x^2\, b_y^2 = 0;\quad (a_x^2\, b_y^2,\ a_x^2\, b_y^2)^{1,1} \equiv c_x^2\, d_y^2 = 0,$$

the relation between the forms $a_x^2\, b_y^2$; $c_x^2\, d_y^2$ being reciprocal in that each differs only by a constant factor from the (1, 1) transvectant from the other.

If now, p be (x, y), $\Gamma_1(p)$ be (x', y'), and $\Gamma_2(p)$ (x'', y''), then the above construction reduces to:

(5.5)
$$\begin{bmatrix} a_x\, a_{x'}\, b_y^2 = 0 \\ c_x^2\, d_y\, d_{y'} = 0 \end{bmatrix},\quad \begin{bmatrix} c_x\, c_{x''}\, d_y^2 = 0 \\ a_x^2\, b_y\, b_{y''} = 0 \end{bmatrix}.$$

These equations express $(x'\ y')$, $(x''\ y'')$ rationally in terms of (x, y), and are the double-binary equations of Γ_1, Γ_2.

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* If t is known, the locus of s is a twisted cubic which cuts l *only once*.

Invariant curves on Q_3.

The simplest curves on Q_3, invariant for Γ_1, Γ_2 are of three types. By (2.5), the ∞^2 quartic curves cut out on Q_3 by fundamental quadrics of the second system are all invariant for Γ_1. Similarly, the intersections of Q_3 with quadrics of the first system are invariant for Γ_2. These two systems of curves (which, be it noticed, are not fundamental curves) may be called *the first and second types of invariant nets on Q_3*. The intersections of Q_3 with other quadrics of the same system form a pencil of fundamental curves, invariant for both Γ_1 and Γ_2; we call this *the third type of invariant curves.*

In particular, the conjugate fundamental curves γ, γ' contained in Q_3 are invariant curves of the third type.

(5.6) *The locus of nodes of invariant curves of the first and second types is the pair of conjugate fundamental curves γ, γ' contained in Q_3.*

For, an invariant curve of, say, the first type, has a node p when the quadric of the second system which contains it has contact with Q_3 at p. Now, the tangent at p to the fundamental curve of the second system through it is $p\,\Gamma_2(p)$ (2.6). Hence we can find a fundamental quadric Q_2 to touch any assigned plane through $p\,\Gamma_2(p)$ at p. Hence we can find a Q_{12} to touch the Q_3 at p, if and only if, $p\,\Gamma_2(p)$ is tangent to Q_3. But Q_3 is invariant for Γ_2. Hence $p\,\Gamma_2(p)$ can be tangent to Q_3, only if $p\,\Gamma_2(p)$ is a generator, or if p is at one of the points $B_0B_1B_2B_3$, for which $p \equiv \Gamma_2(p)$. Thus, by (5.3), the locus of p is the pair of conjugate curves γ, γ'.

We may identify the invariant curve of the first type, which has a double point at any assigned point p of γ or γ'. It has been shewn that the quadric Q_2 containing the invariant curve, has the same tangent plane at p as Q_3; hence it must contain two of the three complex-lines through p in this plane, namely, the lines $p\,\Gamma_2(p)$, $p\,\Gamma_3(p)$, $p\,\Gamma_1(p)$. If Q_2 were to contain $p\,\Gamma_2(p)$, then the invariant curve would consist of $p\,\Gamma_2(p)$ and a cubic curve which cuts it in a pair of Γ_1; thus the

invariant will *not* have a double point at p. Hence Q_2 must contain $p\,\Gamma_3(p)$ and therefore also, by (5.3), $p\,\Gamma_1(p)$. Thus the invariant curve consists of the line $p\,\Gamma_1(p)$ and the twisted cubic which is its transform in Γ_1. In particular, if p is at one of the points B_r, the invariant curve determined in this way becomes the twisted cubic $A_0A_1A_2A_3B_r$ and its tangent-generator at B_r (compare 5.1).

However, if p is at B_r, the line $p\,\Gamma(p)$ is indeterminate, and so the invariant-curve is not determined uniquely, but in ∞^1 ways. That is, the intersections of Q_3 with the ∞^1 cones Q_2 with vertex B_r, are the invariant curves in question. The nodal tangents of these curves would constitute pairs of an involution, the double members of which are, from what has preceded, the two generators through B_r. Hence :

(5.7) *The invariant curve of the first type with an assigned double point p is in general unique, and is composed of a twisted cubic and a generator. If p is B_0, B_1, B_2 or B_3, we have however a pencil of invariant-curves, which are in general irreducible, and whose nodal tangents separate harmonically the generators through the node.*

When p is at A_r, the invariant curve is easily seen to consist of the two generators through A_r and the conic $A_p A_q A_s$. The theorem (5.7) shews that the invariant curves with a node at one of the points B_r may be unambiguously characterised as *the irreducible nodal invariant curves of the first type.*

§ 6. The Fundamental Congruences.

Considering the invariant quadric Q_3, we denote the congruence of lines joining corresponding pairs of Γ_r on Q_3 by Q_{3r}, and call it *a fundamental congruence of the cubic complex* Γ $(r = 1, 2)$. Thus Q_3 determines two fundamental congruences Q_{31}, Q_{32}.

If p is any point on Q_3, then p, $\Gamma_1(p)$, $\Gamma_2(p)$ all lie on a fundamental quartic curve, namely, an invariant curve of the third type on Q_3. Now, the joins of pairs of Γ_1, Γ_2 on a fundamental curve of the third system are all generators of the quadric which contains the curve

and its conjugate (1.2). Thus the congruences Q_{31}, Q_{32} are composed respectively of the two reguli of the quadrics which contain a fundamental curve on Q_3 and its conjugate curve—that is to say, of the quadrics which belong to the rational cubic series determined by Q_3. The congruence of the two reguli of quadrics of the cubic series were mentioned in § 4. Restating the results :

(6.1) Each of the fundamental congruences Q_{31}, Q_{32} is of order-class (3,3) and is its own polar reciprocal congruence in regard to Q_3.

If Γ' is the polar complex of Γ in regard to Q_3, then Γ and Γ' should have a (9,9) congruence in common: this (9,9) congruence is seen to consist of the fundamental congruences Q_{31}, Q_{32}, and the three linear congruences determined by pairs of opposite edges of Δ_3, which, by (2.9), belong to Γ.

A fundamental property of the congruences Q_{31}, Q_{32} is given by the theorem :

(6.2) *The two sets of three lines which lie in an arbitrary plane P, and belong respectively to the two congruences Q_{31}, Q_{32} form two mutually polar triangles in respect of the section of Q_3 by P. The dual theorem is also true.*

For, let P cut the three nets of fundamental quadrics in the three nets of conics N_1, N_2, N_3; let the intersection of P with Q_3 be called n_3. It follows from (3.4), that n_3 contains three pairs p_1p_1', q_1q_1', r_1r_1' of Γ_1, and three pairs p_2p_2', q_2q_2', r_2r_2' of Γ_2 and that p_1p_1' is harmonic to q_2q_2' and r_2r_2', etc. In other words the two triangles $(p_1p'_1\ q_1q_1',\ r_1r_1')$, $(p_2p_2',\ q_2q_2',\ r_2r_2')$ are polar triangles with respect to n_3. But these lines are clearly the lines of Q_{31}, Q_{32} which lie in the plane P. Hence the theorem.

(6.3) *If a line l_1 of Q_{31} meets a line l_2 of Q_{32}, then either l_1 and l_2 pertain to the same quadric of the rational cubic series, or each of l_1, l_2 meets also the polar line of the other in regard to Q_3.*

For, let the plane of l_1, l_2 be the plane P of the last theorem, and let l_1 be p_1p_1'. Then l_2 may be either p_2p_2' or one of the lines $q_2q'_2$, r_2r_2'. In the latter case, it follows from the harmonic property, that each of l_1, l_2 meets the polar line of the other. Also, from the proof of (3.4) it is seen that p_1, p_1', p_2, p_2' are base-points of a pencil of conics N_3. Hence p_1p_1', p_2p_2' is a line-pair of N_3, so that there is a quadric of the third system, which contains p_1p_1', p_2p_2'; since these are congruence-lines, the quadric belongs to the cubic series. Thus, when l_2 is $p_2p'_2$, l_1 and l_2 are generators of the same quadric of the rational cubic series.

From these theorems, we may deduce an important result :

(6.4) *If the three lines of Q_{3r} through any point p intersect the polar plane of p with respect to Q_3, in p_1, p_2, p_3, then p_2p_3, p_3p_1, p_1p_2 are lines of Q_{3s}. $(r, s = 1, 2)$.*

This follows immediately from (6.2), by taking successively the planes pp_2p_3, pp_3p_1, pp_1p_2 for the plane P. Since p_2p_3 meets not only pp_2 and pp_3, but also evidently their polar lines, it follows from (6.3) that p_2p_3 is that line of Q_{3s} in the plane pp_2p_3, which does *not* belong to the same quadric of the cubic series as either pp_2 or pp_3.

(6.5) *The three lines of Q_{31} through any point p are met respectively by the three lines of Q_{32} through $\Gamma_1(p)$; and the plane of the points of intersection is the polar plane of $\Gamma_1(p)$ in regard to Q_3.*

For, the quadrics of the cubic series K are fundamental quadrics of the third system and therefore invariant for Γ_1, Γ_2. Therefore the three quadrics K which pass through p also pass through $\Gamma_1(p)$; thus, the three lines of Q_{31} through p, and the three lines of Q_{32} through $\Gamma_1(p)$ can be paired into generators of opposite systems of these three quadrics,— which proves the first part.

To prove the second part, let K be one of the three quadrics of the cubic series, which contains p, $\Gamma_1(p)$. The generator of K which belongs to Q_{32} and which passes through $\Gamma_1(p)$ will cut Q_3 in a pair of

Γ_2—say, in $q\ \Gamma_2(q)$. Also, let the same generator cut the generator of the other system through p, in p'. Thus $\Gamma_1(p)\ p'$ is a pair of points on a generator of the fundamental quadric K of the third system, with the property that $\Gamma_1(\Gamma_1(p)) = p$ lies on the generator of the other system through p'. Hence, by (5.4), $p'\ \Gamma_1(p)$ is harmonic to $q\ \Gamma_2(q)$, that is to say, p' and $\Gamma_1(p)$ are conjugate points in regard to Q_3. Similarly each of the three intersections is conjugate to $\Gamma_1(p)$, so that the plane of the intersections is the polar plane of $\Gamma_1(p)$.

If, whenever a congruence contains the six edges of a **tetrahedron,** we agree to say for short, that it *contains* the tetrahedron, it is easy to see that the preceding two theorems point to the existence of ∞^3 tetrahedra contained in a fundamental congruence. For, if the three lines of Q_{31}, through any point p are cut by the polar plane of $\Gamma_1(p)$ in $p_1\,p_2\,p_3$, then, by (6.5), $\Gamma_1(p)p_1$. $\Gamma_1(p)p_2$, $\Gamma_1(p)p_3$ are the three lines of Q_{32} through $\Gamma_1(p)$. Then, replacing p by $\Gamma_1(p)$ in theorem (6.4), we see that p_2p_3, p_3p_1, p_1p_2 must all be lines of Q_{31}. Thus the three lines of Q_{3r} through any point, and the residual lines of Q_{3r} contained in the planes of every two of them are the six edges of a tetrahedron. Thus we have:

(6.6) *A fundamental congruence Q_{3r} contains ∞^3 tetrahedra, each of which is uniquely determined by assigning a vertex or a face. A tetrahedron contained in Q_{3r} has the property, that pole of any face with respect to Q_3 is the point which corresponds to the opposite vertex in Γ_r $(r = 1, 2)$.*

Concerning a tetrahedron of Q_{3r} and the tetrahedron formed by the points which correspond to its vertices in Γ_r, we can state:

(6.7) If $p_1p_2p_3p_4$ be a tetrahedron of Q_{3r}, then the tetrahedron $\Gamma_r(p_1)\ \Gamma_r(p_2)\ \Gamma_r(p_3)\ \Gamma_r(p_4)$ is (1) the polar tetrahedron of $p_1p_2p_3p_4$ with respect to Q_3, (2) also a congruence tetrahedron of Q_{3r} $(r = 1, 2)$.

The first part follows from (6.6); the second part follows from the fact that Q_{3r} is its own polar reciprocal congruence with respect to Q_3.

The two tetrahedra of (6.7) are further related in that the four lines which join their corresponding vertices are complex-lines, while the twelve lines which join their non-corresponding vertices are, by theorem (6.4), lines of Q_{3r}.

It may be noticed, that when the vertex p of a congruence tetrahedron of Q_{3r} is taken on Q_3, the tetrahedron is that which has for edges, the two generators through p, the two generators through $\Gamma_r(p)$, the line $p\,\Gamma_r(p)$ and its polar line $(r = 1, 2)$.

We shall deal now with some important consequences which follow from the existence of congruence tetrahedra.

The Quartic Surface ϕ_4.

We define ϕ to be the locus of points p, which are such that $p\,\Gamma_1(p)$ are conjugate points in regard to Q_3.

Since Q_3 is a fundamental quadric of the third system, $p\,\Gamma_3(p)$ are always conjugate points in regard to it. Further we know, by (2.3), that $\Gamma_1(p)$, $\Gamma_2(p)$ $\Gamma_3(p)$ are collinear. Hence if p is conjugate to $\Gamma_1(p)$, it must also be conjugate to $\Gamma_2(p)$; that is, ϕ is equally the locus of points p which are conjugate to $\Gamma_2(p)$ in regard to Q_3.

Further, any line of Q_{31} cuts Q_3 in a pair p, $\Gamma_1(p)$ of Γ_1, and being a complex line contains a pair of Γ_2, say $q\,\Gamma_2(q)$. By (2,7) q, $\Gamma_2(q)$ separate $p\,\Gamma_1(p)$ harmonically, and are therefore conjugate with respect to Q_3, so that they pertain to the locus. Hence *ϕ may be also defined as the locus of pairs of Γ_2 on lines of Q_{31}, and similarly, as the locus of pairs of Γ_1 on lines of Q_{32}.*

Now the lines of Q_{31}, Q_{32} are arranged as reguli of the ∞^1 quadrics K of the rational cubic series defined by Q_3. The locus of these pairs on any particular quadric K is clearly from (5.3), that one of the conjugate fundamental curves on K which is not contained in Q_3. Hence *ϕ is the locus of quartic curves conjugate to the fundamental curves on Q_3.*

Hence, if we represent fundamental curves by points in a plane π, and use the notation of § 4, the locus in π, corresponding to ϕ, is the locus of points which correspond, in the Cremona transformation defined by the quadrangle $\lambda_0\lambda_1\lambda_2\lambda_3$, to the points of the line Q_3; this locus is obviously a conic which passes through the pair of conjugate points on Q_3, and is circumscribed to the harmonic triangle $h_1h_2h_3$. Hence:

(6.8) The locus ϕ is a quartic surface containing the twelve edges of the two tetrahedra Δ_1, Δ_2, and the two conjugate fundamental curves γ, γ' on Q_3.

Since ϕ is the locus of pairs of Γ_2 on lines of Q_{31}, it follows that the pair of Γ_2 on any line of Q_{31} gives two of its intersections with ϕ; the question as to the nature of the two residual intersections is answered by a closer examination of congruence-tetrahedra.

Let p_1 be any point on the line l of Q_{31}. The unique congruence tetrahedron of Q_{31} determined by the vertex p_1 has a second vertex p_2 on l. The pairs of points p_1p_2 are related in mutual symmetry, and must constitute an involution on l. From (6.6), the points p_1p_2 are such that p_1 is conjugate to $\Gamma_1(p_2)$ and p_2 to $\Gamma_1(p_1)$. The double points must be the points q, $\Gamma_2(q)$ which form the pair of Γ_2 on l; for, these points are such that q is conjugate to $\Gamma_2(q)$, and therefore also to $\Gamma_1(q)$. Hence we obtain a theorem which is the generalisation (5.4):

(6.9) A point-pair p_1p_2 which separates harmonically the pair of Γ_r contained in a line of Q_{3s} has the property that p_1 is conjugate to $\Gamma_s(p_2)$, and p_2 to $\Gamma_s(p_1)$ $(r, s = 1, 2)$.

The distinction between Q_{3r}, Q_{3s} which is prominent in this theorem, disappears in (5.4) on account of the fact that each of the reguli of Q_3 belongs to each of the fundamental congruences.

There is however an important point which has been overlooked in Theorem (6.9). Consider in fact the pairs of points p_1p_2 on l, so related that p_2 is conjugate to $\Gamma_1(p_1)$. It is clear that the point p_2 is uniquely determined by p_1; but if p_2 is given, the locus of $\Gamma_1(p_1)$ is a plane, namely, the polar plane of p_2. Since p_1 is on l, the locus of $\Gamma_1(p_1)$ is also a twisted cubic which cuts l in its pair of Γ_1. Thus there are *three* admissible positions of $\Gamma_1(p_1)$ and therefore, of p_1. In

other words, the correspondence $p_1 \rightarrow p_2$ is a (3,1) correspondence; on the other hand, the theorem of the uniqueness of the congruence-tetrahedron with a given vertex assures us that this correspondence must be *effectively* (1,1). To satisfy these two requirements, it is clear that the parameter of p_2 on l must be a rational cubic function of the parameter of p_1, whose numerator and denominator have a quadratic factor in common. Accordingly, there must exist two positions of p_1, for which p_2, defined as above, becomes indeterminate; that is to say, two positions of p_1, for which $\Gamma_1(p_1)$ is conjugate to *every* point on l, and lies therefore, on the polar line of l. These two positions of p_1 must belong to ϕ, and are therefore the residual intersections of l and ϕ. We have thus the theorem:

(6.10) Two of the four intersections of any line l of Q_{3r} with ϕ are the pair of Γ_s contained in l; the two residual intersections have the property, that the points corresponding to them in Γ_r lie on the polar line l' of l, and constitute in fact the residual intersections of l' with ϕ $(r, s = 1, 2)$.

From (6.9), it follows that

(6.11) the pairs of Γ_s situated on the edges of a tetrahedron of Q_{3r} are harmonically separated by the vertices of the tetrahedron $(r, s = 1, 2)$,

A further property of the residual intersections of a congruence-line with ϕ may be obtained. Let K be the quadric of the cubic series which contains the line l of Q_{31} K cuts ϕ in two fundamental curves $\varepsilon, \varepsilon'$, of which one—say ε—contains the pair of Γ_2 on l, and therefore the other, ε', contains the residual intersections of l and ϕ. The generator l of K is cut in pairs of an involution by the pencil of fundamental curves contained in K; two pairs of this involution are the pairs of Γ_1, Γ_2 which are cut out by the conjugate fundamental curves (5.3). From the harmonic property of pairs of $\Gamma_1, \Gamma_2, \Gamma_3$ contained in a complex-line, it follows that the pair of double points of the involution is the pair of Γ_3 contained in l. Hence

(6.12) the residual intersections of a congruence-line l with ϕ separate harmonically the pair of Γ_3 contained in l.

It is clear that the surface ϕ (which is invariant for Γ_1, Γ_2) might be used as a basis for the study Γ_1, Γ_2, in the same way as we have used Q_3; only, there will be this difference, that while a line of Q_{3r} cuts Q_3 in a pair of Γ_r, it will cut ϕ in a pair of Γ_s $(r, s = 1, 2)$. If p is a point on ϕ, then $p\,\Gamma_2(p)$ is a line of Q_{31} and is algebraically distinct from the two other lines of Q_{31} through p; let us call $p\,\Gamma_2(p)$ *the odd line* of Q_{31} through p. Since p, $\Gamma_1(p)$, $\Gamma_2(p)$ are on a fundamental curve the two lines $p\,\Gamma_1(p)$, $p\,\Gamma_2(p)$ are generators of the same fundamental quartic (4.2); thus, the odd lines of Q_{31}, Q_{32} through p pertain to the same quadric of the cubic series. Hence, by the dual of (6.2), the polar plane of $p\,\Gamma_1(p)$ in regard to the tangent cone to Q_3 from p is the plane of the two lines of Q_{31} through p, other than $p\,\Gamma_2(p)$. But since p is on ϕ, $\Gamma_1(p)$ is conjugate to p; hence this plane is the polar plane of $\Gamma_1(p)$ in regard to Q_3. Thus:

(6.13) If p is a point on ϕ, the polar plane of $\Gamma_1(p)$ is the plane of the two lines of Q_{31} through p_1 other than the odd line $p\,\Gamma_2(p)$.

Degenerate Congruence-tetrahedra.

It is seen from (6.6) that a congruence-tetrahedron is in general uniquely determined by assigning a vertex p. We say *in general*, as this statement would naturally cease to be true if p is at one of the singular points of Q_{31}, Q_{32}, namely, one of the vertices of Δ_1, or Δ_2. The lines of Q_{31}, through a vertex A_r of Δ_1 form a quadric cone whose base is the section of Q_3 by the opposite face; the lines of Q_{32} through A_r are tangent to Q_3. Thus in these cases the congruence-tetrahedron would be indeterminate.

When the vertex p is in general position, the congruence tetrahedron determined by it is a *proper* tetrahedron. We shall shew that when p is on ϕ, the congruence-tetrahedron degenerates in a certain definite manner. Indeed it will appear that *the surface ϕ may also be defined as the locus of double vertices of degenerate congruence-tetrahedra.*

Let the lines of Q_{31} through the point p on ϕ be $p\,\Gamma_2(p)$, pq, pr. These three lines are the edges of the congruence-tetrahedron whose

vertex is p; by (6.7), the three other vertices of the tetrahedron are the inter-sections of these edges with the polar plane of $\Gamma_1(p)$. But by (6.13), the polar plane of $\Gamma_1(p)$ is the plane pqr; thus the tetrahedron degenerates, so as to have a double vertex, the two other vertices lying on pq, pr respectively. To examine whether these two vertices have definite limiting positions on pq, pr, we remember that p is a residual intersection of the congruence-line pq with ϕ. Hence, if $p_1 p_2$ are the residual intersections of $p\Gamma_2(p)$ with ϕ, and p_1', p_2' their harmonic conjugates with respect to $p\Gamma_2(p)$, the aspect of the tetrahedron to the line pq is the same as the aspect of the congruence tetrahedron whose vertex is p_1, to the line $p\Gamma_2(p)$. But from (6.9), (6.11), it follows that $p_1 p_1'$ are vertices of the same congruence-tetrahedron. Hence we conclude that the congruence-tetrahedron with a vertex p on P has a double vertex at p and two other definite vertices situated on the two congruence-lines through p other than the odd line.

Let $p\Gamma_2(p)$ cut Q_3 in $s\Gamma_1(s)$; it was seen in (6.12) that the three pairs $p_1 p_2$, $p\Gamma_2(p)$, $s\Gamma_1(s)$ belong to the involution whose double points are the pair of Γ_3 contained in the line. Further, $p_1' p_2'$, are by construction the harmonic conjugates of $p_1 p_2$ in regard to $p\Gamma_2(p)$; hence $p_1 p_2 p_1' p_2'$ form a syzygetic tetrad with respect to the three mutually harmonic pairs of Γ_1, Γ_2, Γ_3 contained in the line. Further, p_1' is a single vertex of a oongruence-tetrahedron whose double vertex is p_1; hence two of the three congruence-lines through p_1' coincide with $p_1'p$. That is to say, p_1' and similarly p_2' are *the focal points* on the congrgence-line. We have thus reached the theorem :

(6.14) The two focal points on a line l of a fundamental congruence form, together with the residual intersections of l with ϕ, a syzygetic tetrad in respect of the three pairs of Γ_1, Γ_2, Γ_3 contained in l; hence, by (2.10), they are cut out on l by one and the same desmic surface.

It also follows from this that $(p_1' p_2)$, $(p_2' p_1)$ are pairs of conjugate points in regards to Q_3.

This theorem may also be investigated by the method of point-representation of fundamental quartic curves explained in § 4. Let qq'

be the conjugate points with respect to the quadrangle $\lambda_0\lambda_1\lambda_2\lambda_3$, situated in the line Q_3. K is the curve of the third class enveloped by the joins of points on Q_3 to their conjugates; ϕ is the conic-locus of points conjugate to points on Q_3. Let r_1 be a point on Q_3, r_2 its conjugate on ϕ; let r_1r_2 cut ϕ again in p and touch its envelope K at p'. Let $r_1'r_2'$ be consecutive positions of r_1, r_2 on Q_3 and ϕ respectively. Then the quadrilateral $(r_1r_1', r_1r_2', r_2r_1'. r_2r'_2)$ is such that two pairs of its opposite vertices (r_1r_2, r_1r_2') are conjugate points in regard to $\lambda_0\lambda_1\lambda_2\lambda_3$; hence the third pair of vertices must also be conjugate points. One of the remaining vertices lies clearly on Q_3; the other remaining vertex, which is, in the limit, situated on r_1r_2, must be its conjugate point, and must therefore lie on ϕ. Hence this latter vertex must be p. Also, the point p', being the limit of the intersection of r_1r_2, $r_1'r_2'$ must be the harmonic conjugate of p with respect to r_1r_2, from the property of the complete quadrangle $r_1r_2r_1'r_2'$. Thus (pp', r_1r_2) is a harmonic range. This result when interpreted, means that the two conjugate fundamental curves on a quadric K of the cubic series, separate harmonically the curve of contact of K with its envelope F_8, and the curve, other than one of the conjugate fundamental curves, in which K intersects the quartic surface ϕ. Replacing this harmonic relation between the four curves by the corresponding relation between the point-pairs which they cut out on a generator of K, we are easily led to the result of (6.14).

Further, the fact that (pp', r_1r_2) is a harmonic range, shows that there exists a conic D passing through $\lambda_0\lambda_1\lambda_2\lambda_3 pp'$. In this way, a unique conic D corresponds to each line of the cubic envelope, while to the conic D corresponds the tangent to the envelope at any of its intersections with D. This when interpreted, leads to the following theorem :

(6.15) Let a desmic surface D cut the focal surface F_8 in the quartic curves γ_r $(r = 1, 2 \dots 8)$. Then the generators of the eight quadrics of the cubic series which touch F_8 along the curves γ_r, are cut by the desmic surface D in their focal points and residual intersections* with ϕ.

* The term 'residual intersections of a line of Q_{3r} with ϕ' denotes the intersections other than the pair of Γ_s contained in the line $(r, s = 1, 2)$.

A congruence line is bitangent to the focal surface at its two focal points; its four residual intersections with F_8 are seen to be cut out by two fundamental curves on the quadric of the cubic series which contains the line. The pencil of fundamental curves cuts out on any generator an involution containing the pairs of Γ_1, Γ_2 which are cut out by the two conjugate fundamental curves. Hence the four residual intersections with F_8 form two pairs harmonic to the pair of Γ_3 contained in the line.

The Transformation Γ_3.

It was shown in (6.2) that the two triads of lines of Q_{31}, Q_{32} which lie in a plane π form two polar triangles in respect of the section of Q_3; they are therefore in perspective, the centre of perspective being (say) P'. Further, if P be the pole of π with respect to Q_3, then (6.4) and (6.5) shew that the joins of P to the vertices of the two polar triangles in π, are respectively the lines of Q_{32}, Q_{31} which pass through P. It is also easy to see that the lines joining P to *corresponding* vertices of the two triangles are congruence-lines pertaining to the *same* quadric of the rational cubic series. From the perspective, the planes of the three such pairs of lines all contain PP', so that PP' is the common tangent to the three quadrics of the cubic series, which pass through P. Hence PP' is the tangent at P to a fundamental curve of the third system, so that $\Gamma_3(P)$ lies in PP' (2.6). But P, $\Gamma_3(P)$ must be conjugate points in regard to Q_3 so that $\Gamma_3(P)$ also lies in π. Hence:

(6.16) The pole, in regard to Q_3, of any plane, and the centre of perspective of the two congruence-triangles contained in the plane, are correspondents in Γ_3.

Denote now by ψ_1, ψ_2 the locus of points p such that $p\,\Gamma_3(p)$ belongs to Q_{31}, Q_{32} respectively. Thus ψ_1, ψ_2 are the locus of pairs of Γ_3 situated on lines of Q_{31}, Q_{32} respectively. It follows from the perspective property (6.16), that if p be a point on ψ_1, the congruence-triangles of Q_{31}, Q_{32} in the polar plane of p are such that two of the vertices lie the non-corresponding sides of the latter; in other words two of the lines of Q_{32} in the polar plane cut out harmonic pairs on the section of Q_3. Hence also two of the lines of Q_{32} through p are conjugate lines in respect of Q_3. Thus:

(6.17) ψ_r is also the locus of points p which are such that two of the three lines of Q_{3s} through p are conjugate lines $(r, s = 1, 2)$.

The two lines of Q_{3s} of this theorem are clearly the two lines other than that which corresponds to the line $p\Gamma_3(p)$ of Q_{3r}. To put the matter in a different way, we can also say that ψ_r is the locus of points p, which are such that the plane of two of the lines of Q_{3r} through p contain a line of Q_{3s} through p. This statement as well as (6.17), have however to be corrected for the case in which p is on the quadric: for this case, two of the lines of Q_{3s} through p are generators and therefore each conjugate to the third line, but it is not true that $p\Gamma_3(p)$ is a line of Q_{3r}. As a matter of fact, the argument which led up to (6.17) fails here, since two of the vertices of one congruence-triangle lie in the *corresponding* sides of the other, so that p is not the centre of perspective.

We have next to determine the order of the loci ψ_r.

Correspondences between two polar congruence-lines.

Let l, l' be polar lines with respect to Q_3, both belonging to the congruence Q_{31}. Let them cut Q_3 in $p\Gamma_1(p)$, $p'\Gamma_1(p')$ respectively and let the pairs of Γ_2 Γ_3 contained in them be $q\Gamma_2(q)$, $r\Gamma_3(r)$ $q'\Gamma_2(q')$, $r'\Gamma_3(r')$ respectively.

Through any point λ_1 on l there pass two lines of Q_{32} not belonging to the same quadric of the cubic series as l; by (6.3), these meet l' in μ_1, μ_2 (say). Thus the congruence Q_{32} establishes a (2,2) corresdence $C_{2,2}$ between l, l'. When λ_1 is at p, μ_1, μ_2 are p', $\Gamma_1(p')$ respectively; and the two points on l corresponding to either of p', $\Gamma_1(p')$ are the same two points p, $\Gamma_1(p)$. Hence the (2, 2) correspondence is *sub-rational*; that is, if $\mu_1\mu_2$ on l' correspond, in $C_{2,2}$, to any point λ_1 on l, then each of μ_1, μ_2 correspond to the *same* two points λ_1, λ_2 on l. It is clear that $\lambda_1\lambda_2$, $\mu_1\mu_2$ are pairs of two involutions on l, l', and the correspondence $C_{2,2}$ may be regarded as a homography between the pairs of the two involutions.

From (6.7), we learn that if $p_1p_2p_3p_4$ be a tetrahedron of Q_{31},

$$\Gamma_1(p_1),\ \Gamma_1(p_2),\ \Gamma_1(p_3),\ \Gamma_1(p_4),$$

its polar tetrahedron, then $p_1 p_2$ and $\Gamma_1(p_3)$, $\Gamma_1(p_4)$ are polar congruence-lines possessing the property that

$$p_1 \Gamma_1(p_3),\ p_1 \Gamma_1(p_4),\ p_2 \Gamma_1(p_3),\ p_2 \Gamma_1(p_4)$$

are all lines of Q_{32}. Hence we conclude that the involution $\lambda_1 \lambda_2$ is simply the involution of vertices of congruence-tetrahedra, and has therefore the double points $q\,\Gamma_2(q)$ (6.11); similarly $\mu_1 \mu_2$ is a pair of the involution whose double-points are $q'\,\Gamma_2(q')$. We notice also that when $\lambda_1 \lambda_2$ are given, $\mu_1 \mu_2$ may be defined as the correspondents in Γ_1 of the other vertices of the congruence-tetrahedron, two of whose vertices are $\lambda_1 \lambda_2$.

It thus appears that the two lines of Q_{32} not corresponding to l, which lie in any plane $(l\mu_1)$ through l, both pass through μ_1 and are the lines $\mu_1 \lambda_1$, $\mu_1 \lambda_2$. Let the third line of Q_{32} in this plane intersect l in λ. Then for each λ correspond two points $\lambda_1 \lambda_2$; also, for a given λ_1 there correspond two points λ, λ', for we can take either of the planes (μ_1), $(l\mu_2)$. Since (λ_1, λ_2) is a pair of a definite involution, it follows that this (2,2) correspondence must be subrational, so that λ, λ' should also belong to an involution. Also λ, λ' will coalesce when μ_1 and μ_2 coalesce, that is, when $\mu_1 = q'$ or $\Gamma_2(q')$; the corresponding positions of $(\lambda_1 \lambda_2)$ are the two residual focal pairs on l, harmonic to q, $\Gamma_2(q)$.

It follows that there must be four positions of λ, for which it is conjugate to one of its corresponding points λ_1, λ_2; now, if λ is conjugate to λ_1, then one out of the two lines of Q_{32}, $\lambda_1 \mu_1$, $\lambda_1 \mu_2$ will the polar line of one of the similar lines of Q_{32} through λ, so that the line of Q_{32} through λ corresponding to l will be conjugate to one of the other lines. Thus, the four such points λ together with $r\,\Gamma_3(r)$ will be the intersections of l with ψ_1. Thus ψ_1 and similarly ψ_2 are sextic loci invariant for Γ_3. It is also clear from the definition of these surfaces, that they are both circumscribed to the three fundamental tetrahedra Δ_1, Δ_2, Δ_3, and that ψ_1 touches Q_3 at the vertices of Δ_2 and has double points at the vertices of Δ_1, the correspnding tangent-cones being those which contain the sections of Q_3 by the opposite faces From the invariance with respect to Γ_3, it follows that ψ_1 and ψ_2 contain the edges of Δ_3 and have triple points as its vertices.

The lines of the cubic complex Γ which meet l and l' establish a (3,3) correspondence between the points of the two lines. This correspondence must include as part of itself the sub-rational (2,2) correspondence $C_{2,2}$ determined by the lines of Q_{32}. Hence *the transversals of l, l' which are complex-lines, but not lines of Q_{32}, establish a homography between l, l' and therefore generate a quadric regulus.*

Further, since Q_{31}, is a (3,3) congruence, there must be six lines of Q_{31} which meet l, l', or two lines of Q_{31} other than the four generators of the quadric Q_3. It is clear that these two lines must be polar lines, and therefore, the diagonals of the quadrilateral which they form with l, l', must be lines of Q_{32}. If these lines intersect l in the conjugate points s, s', then obviously s, s' are two of the six intersections of l with ψ_2; these two intersections distinguish themselves algebraically from the remaining four, in that l itself is one of the two conjugate lines of Q_{31} at these points.

The locus of pairs of Γ_3 on generators of the quadric Q_3 consists of a pair of curves whose partial orders are (2,4) and (4,2). These two curves together constitute the complete intersection of Q_3 with ψ_1, as well as with ψ_2. The double-binary forms which represent these curves on Q_3 are the Jacobians with respect to one of the variables, of the two (2,2) forms representing the conjugate fundamental curves (γ, γ') on Q_3,*

§ 7. The Generalised Isogonal Transformation in a Space with a Projective Metric.

Let S be a space with a Cayleyan metric defined by the Absolute quadric Q_3. A cubic transformation Γ_1 of the type we have discussed, will be called *a generalised isogonal transformation of* S, if it leaves invariant the Absolute quadric Q_3. Writing Γ_1 in its desmic form :

$$\Gamma_1 = \{\Delta_1, (\Delta_2, \Delta_3)\},$$

so that Δ_1 is the singular tetrahedron of Γ_1, it follows that the invariant quadric Q_3 must be circumscribed to the singular tetrahedron Δ_1, and

* Compare H. W. Turnbull cited in the Introduction.

also to one of the two other tetrahedra Δ_2, Δ_3—say, to Δ_2. It also follows that the third tetrahedron Δ_3 must be a self-polar tetrahedron of the Absolute. Thus:

(7.1) The singular tetrahedron of an isogonal transformation, together with *one* of the two other tetrahedra of the desmic system defining it, must be inscribed in the Absolute; the third tetrahedron of the system is consequently self-polar in regard to the Absolute.

Let us shew next that an isogonal transformation is in general determined uniquely by assigning its four singular points arbitrarily on the absolute. For this we have to shew that Δ_2 and therefore also Δ_3 are completely determined when Δ_1 is known. Let the equation of Q_3, with respect to the known tetrahedron Δ_1 for tetrahedron of reference, be:

$$px_0x_1 + p'x_2x_3 + qx_0x_2 + q'x_3x_1 + rx_0x_3 + r'x_1x_2 = 0.$$

Then it is an easy piece of algebra to shew that there exists effectively only *one* substitution of the form:

$$x_0 = \lambda_0 x'_0, \;\; x_1 = \lambda_1 x'_1, \;\; x_2 = \lambda_2 x'_2, \;\; x_3 = \lambda_3 x'_3,$$

which transforms this equation into one of the form:

$$p''(x'_0x'_1 + x'_2x'_3) + q''(x'_0x'_2 + x'_3x'_1) + r''(x'_0x'_3 + x_1'x'_2) = 0.$$

This shews, in substance, that there is only one other inscribed tetrahedron of the quadric, which is desmic to Δ_1. Hence:

(7.2) A tetrahedron Δ_1, inscribed to a quadric Q_3 is desmic to only one other inscribed tetrahedron Δ_2; also, it is desmic to only one tetrahedron Δ_3 self-polar in regard to Q_3. The three tetrahedra Δ_1, Δ_2, Δ_3 form a desmic system.*

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* If S is a Euclidean space, and the quadric Q_3 a sphere, then the vertices of the tetrahedron Δ_3 thus determined, are, what have been called 'Lemoine points of the tetrahedron Δ_1' by Mr. V. Ramaswami Aiyar, in a paper read before the Fifth Conference of the Indian Mathematical Society. The normal co-ordinates of the vertices of Δ_3, with reference to Δ_1, are (according to him) proportional to the circum-radii of the faces of the latter.

The actual determination of Δ_2 from Δ_1 can be effected by theorems of previous sections; for instance, by (5.1), or (5.2). The metrical relation may be derived from the theorem :

(7.3) The bisectors of the angle between any two faces of Δ_1 are the two desmic planes through their intersection.

For, it follows immediately from the property of the desmic system, that the two desmic planes through any edge of, say, Δ_1, separate harmonically the two faces of Δ_1 through the edge. Further, the pair of desmic planes through an edge of Δ_1 is a fundamental quadric of the first system (1.3). Now every quadric of the first system is circumscribed to the self-polar tetrahedron Δ_3 of Q_3, and is therefore outpolar to Q_3 (and in fact, to any quadric of the second or third system). Two planes can constitute an out-polar quadric of Q_3, only if they are conjugate with respect to it. Thus the two desmic planes in question not only separate the two faces of Δ_1 harmonically, but are also conjugate in regard to the Absolute quadric—which shews that they are the bisectors of the angle between the faces.

From this theorem follows a simple metrical construction for Δ_2. Take the two pairs of bisecting planes of two pairs of faces of Δ_1. Out of the four lines of intersection of these two pairs, two are opposite edges of Δ_3 and are therefore perpendicular; the two remaining non-perpendicular lines are edges of Δ_2 and therefore cut the Absolute in the required vertices.

The preceding sections shew that the properties of the isogonal transformation Γ_1 defined by Δ_1 are inseparably connected with the isogonal transformation Γ_2 of Δ_2. It will be convenient to call Γ_2 '*the co-isogonal transformation of* Δ_1'. The third transformation Γ_3 of the desmic system (which, be it remembered, is not an isogonal transformation) will be called '*the ortho-transformation of* Δ_1.'

The syzygetic system of inscribed tetrahedra.

Before proceeding further, it is desirable to examine the symmetric relation between the pairs of inscribed tetrahedra, such as Δ_1, Δ_2. With certain assumptions we shall shew that it is equivalent to the relation

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between the pairs of tetrads of a binary syzygetic involution, the cross-ratios of which are conjugate complex numbers.

Let Δ_3 be a given self-polar tetrahedron of the Absolute quadric Q_3, so that its equation referred to Δ_3 may be taken in the form :

$$Q_3: \; a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0,$$
$$a_0 + a_1 + a_2 + a_3 = 0.$$

It is clear that there exists a unique inscribed tetrahedron

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3;\ \alpha_0, \alpha_1, -\alpha_2, -\alpha_3;\ \alpha_0, -\alpha_1, \alpha_2, -\alpha_3;$$
$$\alpha_0, -\alpha_1, -\alpha_2, \alpha_3)$$

desmic to Δ_3, with a given vertex $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ on Q_3. Let the family of these inscribed tetrahedra be termed the 'desmic family of Δ_3.' The desmic family is therefore composed of associated pairs of mutually desmic inscribed tetrahedra Δ_1, Δ_2.

Let the two systems of generators on Q_3 be represented by two independent parameters x, y; and let the point of intersection of the generator (x) and the generator (y) be said to have the co-ordinates (x, y). The four involutoric transformations of the points of Q_3 determined by the vertices C_0, C_1, C_2, C_3 of Δ_3 interchange the two systems of generators, while the three mutually harmonic involutions determined by pairs of opposite edges of Δ_3 carry each system of generators into itself. Let the parametric representation (x) be so chosen, that the transformation of x-generators by the three latter involutions is given by the equations

$$x' = -x; \quad x' = \frac{1}{x}; \quad x' = -\frac{1}{x}.$$

Further, let the parametric representation (y) be determined by the requirement that the transformation determined by C_0, between the two systems of generators has the equation

$$x = y.$$

Then it follows that the transformation of y-generators by the three latter involutions, also takes places according to the equations

$$y' = -y; \quad y' = \frac{1}{y}; \quad y' = -\frac{1}{y}.$$

Further, let a point on Q_3 whose co-ordinates (x, y), in the co-ordinate system thus chosen, are conjugate complex numbers, be called a 'real' point. From the above equations, it follows then that a 'real' point is transformed into a 'real' point by any of the involutions determined by Δ_3. That is, if *one* vertex of a tetrahedron Δ_1 of the desmic family is 'real,' then the other vertices of Δ_1, and also the vertices of the associated tetrahedron are 'real.' Let such desmic tetrahedra Δ_1 be termed 'real.'

A 'real' point has, by definition, co-ordinates of the form $x + ix'$, $x - ix'$, where x, x' are real, and can therefore be represented by the complex variable $z = x + ix'$. The vertices of 'real' desmic tetrahedra Δ_1, then correspond to the roots of quartics of the syzygetic pencils:

$$z^4 + 6\lambda z^2 + 1$$

the vertices of the associated tetrahedra Δ_2 corresponding to the conjugate complex numbers If the 'real' tetrahedra Δ_1 are put in correspondence with the points in an Argand plane, whose complex numbers are λ, it is clear that the associated tetrahedra Δ_1, Δ_2 will correspond to points symmetric with respect to the real axis. This fact may be stated in a more general form. The system of 'real' desmic tetrahedra contains three tetrahedra with doubled vertices, which correspond to the intersections of Q_3 with three of the edges of Δ_3. If we put the 'real' tetrahedra in correspondence with the points of an Argand plane, in such wise that *the tetrahedra with doubled vertices correspond to the points* $\lambda_1\ \lambda_2\ \lambda_3$, *then the points corresponding to the associated tetrahedra* Δ_1, Δ_2 *are inverse points in regard to the circle* $\lambda_1\ \lambda_2\ \lambda_3$.*

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* If the 'real' points on Q_3 are put in correspondence with points of the Argand plane, it is clear that the 8 vertices of Δ_1, Δ_2 will correspond to a set of 8 points which are inverted into themselves by the four mutually orthogonal circles $C_0 C_1 C_2 C_3$. A good illustration of this general relation between Δ_1, Δ_2 is furnished by the following result of Mr. V. Ramaswami Aiyar (Question 1358, *J.I.M.S.*, Aug. 1924).

The geometrical property of Δ_1, Δ_2 which is closely connected with this result is: the cross-ratio of generators of one system through the vertices of Δ_1 is equal to the cross-ratio of the generators of the other system through the vertices of Δ_2.

§ 8. The tetracyclic isogonal transformation.

A familiar example of a space possessing a Cayleyan metric is the manifold S of all circles in a projective Cartesian plane π, that is, a plane defined as the aggregate of points, represented uniquely and without repetition by the ratios of three complex numbers. The Absolute quadric which determines the measurement in S is the sub-manifold Q_3 of all point-circles in π. The point-circles $\bar{p}$ which are the elements of Q_3, are in birational correspondence with the points p, which are the elements of π; this correspondence holds for the finite part of the plane π, but has singular elements at infinity. Thus Q_3 and π differ essentially in their mode of closure at infinity; the latter closes along a line, *the line at infinity*, containing two special points, *the circular points*, while the former closes at a point, *the point at infinity*, and along the two isotropics through the point. The 'circular lines' of π correspond birationally to the *isotropics* of Q_3, namely, the generators of Q_3 The quadric manifold Q_3 has been called the '*tetracyclic plane.*' Briefly, the difference between the projective plane π and the tetracyclic plane Q_3 may be said to consist in the fact, that the line at infinity in the former is contracted into a single point,—the point at infinity in the latter—while the circular points of the former are drawn out into the isotropics through the point at infinity in the latter.*

If the inverses of P, in four mutually orthogonal circles with centres at the ortho-centric set ABCD, are P'Q'R'S', then the inverses of P'Q'R'S' in these circles are the same four points PQRS. The centroids of PQRS, P'Q'R'S' are inverse points in the nine-point circle of ABCD.

Here, the syzygetic tetrads PQRS are put in correspond with the ***centroids.*** The centroids $\lambda_1 \lambda_2 \lambda_3$ are at the vertices of the pedal triangle of ABCD, and the result follows from the theorem.

* In connection with the structure of the tetracyclic plane, reference may be made to Coolidge: ***Treatise on the Circle and Sphere***, Chapter IV, and to Woods: ***Higher Geometry***.

It follows from this, that an algebraic curve in π of order $n = r + s + t$, with multiple points of orders r, s at the circular points, will transform through the birational correspondence, into a curve on Q_3 which has a multiple point of order t at the point at infinity, and cuts the two systems of isotropics r and s times respectively. The following table gives the corresponding names of certain simple curves in the three-dimensional, tetracyclic, and ordinary plane geometries:—

Name of curve in three-dimensional projective geometry.	*Name in the geometry of the tetracyclic plane Q_3.*	*Name in Cartesian plane geometry.*
Elliptic Quartic curve on the quadric Q_3.	Cyclic curve.	Bicircular quartic.
Elliptic Quartic curve through the special point.	Cyclic through the point at infinity.	Circular cubic.
Nodal or Cuspidal Quartic curve.	Nodal or Cuspidal cyclic.	Rational Bicircular quartic.
Quartic curve with node, cusp at the special point.	Cyclic with node, cusp at infinity.	General Conic, Parabola.
Quartic curve with node at the special point, the nodal tangents separating the nodal generators harmonically.	... ...	Rectangular Hyperbola.

The generalised isogonal transformation, defined in the last section for a general space possessing a Cayleyan metric, may be called the 'tetracyclic isogonal transformation,' in reference to the particular type of manifold S, whose elements are the circles of a Cartesian plane. The tetracyclic isogonal transformation possesses the following characteristic properties:

(1) It is a circle-transformation, which is also a point-transforma-

tion, that is, which also transforms point-circles, in general, into point-circles.

(2) It possesses four singular circles A_0, A_1, A_2, A_3 all of which are point-circles. The point A_0 corresponds in the transformation to an indeterminate circle orthogonal to the circle $A_1A_2A_3$.

As in the last section, we have to study in connection with the isogonal transformation determined by a point-tetrad Δ_1, also the co-isogonal transformation of Δ_1 (namely, the isogonal transformation defined by the associated tetrad Δ_2), and the ortho-transformation of Δ_1, which is a circle-transformation, which does *not* transform point-circles into point-circles. Before we proceed to study these in detail, let us examine the plane configuration which corresponds to the desmic configuration $\Delta_1, \Delta_2, \Delta_3$.

The vertices $C_0C_1C_2C_3$ of Δ_3 represent four mutually orthogonal circles $C_0C_1C_2C_3$ in the plane; the vertices $A_0A_1A_2A_3$, $B_0B_1B_2B_3$ of Δ_1, Δ_2, are two tetrads of points each of which inverts into the other in each of the circles C_r $(r = 0, 1, 2, 3)$. The notation may be so chosen that (A_rB_0), (A_0B_r) are inverse in C_r. The sections of Q_3 by the twelve desmic planes correspond to twelve circles, which we will call '*the desmic circles.*' These twelve circles are

$$(A_0A_1B_0B_1),\ (A_0A_1B_2B_3),\ (A_0A_2B_0B_2),\ (A_0A_2B_1B_3),\ (A_0A_3B_0B_3),$$
$$(A_0A_3B_1B_2),\ (A_2A_3B_2B_3),\ (A_2A_3B_0B_1),\ (A_3A_1B_3B_1),$$
$$(A_3A_1B_0B_2),\ (A_1A_2B_1B_2),\ (A_1A_2B_0B_3).$$

By (7.3), the twelve desmic circles bisect the angles between the four circles through threes of the four points $A_0A_1A_2A_3$, or of the four points $B_0B_1B_2B_3$, and are orthogonal in pairs. The desmic circles may be also divided into six pairs belonging to the co-axal systems determined by twos of the four circles $C_0C_1C_2C_3$.

If p be any circle, and Γ_1, Γ_2, Γ_3, the isogonal, co-isogonal, and ortho-transformations of Δ_1, we call the circles $\Gamma_1(p)$, $\Gamma_2(p)$, $\Gamma_3(p)$,

the *isogonal circle* of p, the *co-isogonal circle* of p, and the *ortho-circle* of p, respectively. Since p, $\Gamma_3(p)$ are necessarily conjugate points in regard to Q_3, we have the theorem :

(8.1) *A circle is always orthogonal to its ortho-circle.*

It is easy to see that any plane through an edge of Δ_1 is transformed by Γ_1 into another plane through the same edge; pairs of such corresponding planes must be in involution, the double planes being planes which contain self-corresponding points of Γ_1, that is to say, the two desmic planes through the edge. Thus the two planes which pass through an edge of Δ_1, and through a pair of corresponding points of Γ_1 are separated harmonically by the desmic planes through the edge. This gives a construction for the isogonal circle :

(8.2) *Let p be any circle in the plane, p_1 the circle through A_0A_1 orthogonal to p, and p'_1 the circle through A_0A_1 which makes the same angle as p_1 with either of the mutually orthogonal desmic circles through A_0A_1. The six circles of the type p'_1 are all orthogonal to the same circle, namely, the isogonal circle of p.*

The co-isogonal transformation being of the same type as the isogonal, a similar construction carried out with the points $B_0B_1B_2B_3$ obtains the co-isogonal circle. We may obtain the ortho-circle of p, indirectly from its isogonal and co-isogonal circles; for, from (2.3), we see that the isogonal, co-isogonal, and ortho-circles belong to a co-axal system, By (8.1), the ortho-circle of p is that circle of this co-axal system which is orthogonal to p.

A line through a vertex A_0 of Δ_1 is transformed by Γ_1 into another line through the same vertex. Since Q_3 is invariant for Γ_1, it follows that the two lines A_0p, $A_0\Gamma_1(p)$ must cut Q_3 in two points which correspond in Γ_1. Hence :

(8.3) *A pair of isogonal circles of the tetrad $A_0A_1A_2A_3$ has the property of inverting any of the points A_r into two points which are isogonal points of the same tetrad* ($r = 0, 1, 2, 3$).

A similar theorem holds, of course, for co-isogonal circles.

These theorems are of a general character, in so far as the point at infinity in the tetra-cyclic plane does not play any special role in them. We may however expect to derive great advantages from a special choice of the point at infinity. In particular, the choice of a vertex A_0 of Δ_1 for the point at infinity, would put our study *en rapport* with the branch of geometry which is concerned with the properties of a triangle $A_1A_2A_3$. We shall accordingly pay special attention to two types of tetrads, namely, tetrads comprising the point at infinity, and tetrads whose associated tetrads comprise the point at infinity.

(8.4) *The tetrad $B_0B_1B_2B_3$ associated with the tetrad ($A_0 = \infty$, A_1, A_2, A_3) is an orthocentric tetrad, and is in fact composed of the four in-centres of $A_1A_2A_3$ (namely, the centres of the four circles inscribed to the triangle $A_1\ A_2\ A_3$).*

The orthogonality of the two desmic circles through $A_0\ A_1$ is proved in (7.3). When A_0 is the point at infinity, these circles become the *lines* B_0B_1, B_2B_3. Thus any pair of opposite connectors of the tetrad $B_0B_1B_2B_3$ are perpendicular, which shews that it is an orthocentric tetrad. The theorem that the desmic circles bisect the angles between circles through threes of the four points $A_0A_1A_2A_3$ (7.3), shews that the *lines* $A_1\ B_0\ B_1$, $A_1\ B_2\ B_3$ are the internal and external bisectors of the angle $A_2\ A_1\ A_3$.

The theorem shews that an orthocentric tetrad $B_0\ B_1\ B_2\ B_3$ may be defined as one whose associated tetrad comprises the point at infinity. The three remaining points of the associated tetrad are the vertices of the harmonic (or pedal) triangle of $B_0B_1B_2B_3$.

(8.5) *The tetra-cyclic isogonal transformation Γ_1 defined by the tetrad ($A_0 = \infty$, A_1, A_2, A_3) is identical, so far as finite points of the plane are concerned, with the Cartesian isogonal transformation defined by the triangle $A_1\ A_2\ A_3$.*

The theorem follows immediately by using the general construction for the isogonal point (8·2), and remembering that circles through A_0 are

straight lines. In contrast to the conditional identity of the two transformations, their differences should also be observed. These are:

(1) The Cartesian transformation is a point-transformation. The tetra-cyclic transformation is both a circle-transformation and a point-transformation.

(2) The former has three singular points A_1, A_2, A_3 each of these corresponding to the opposite sides; the latter has in addition, the singular point $A_0 = \infty$, which corresponds, for approaches through point-circles, to the circum-circle $A_1A_2A_3$.

In the Cartesian transformation the line at infinity corresponds to the circum-circle. This is understandable in view of the remark that the line at infinity in the Cartesian plane is *contracted* into the point at infinity in the tetra-cyclic plane.

(8.6) *The isogonal transformation of an orthocentric tetrad B_0 B_1 B_2 B_3 carries any point p into the other extremity of the diameter through p to the rectangular hyperbola $B_0B_1B_2B_3$.**

For, let $A_1A_2A_3$ be the pedal triangle of $B_0B_1B_2B_3$, so that the associated tetrad is ($A_0 = \infty$, A_1, A_2, A_3). Then, the rectangular hyperbolas through $B_0B_1B_2B_3$ are cyclic curves with a node at A_0, the nodal tangents being harmonic to the isotropics through A_0. They therefore belong to the class of the curves described at the end of § 5 as *the irreducible nodal invariant curves.* Since they are rational curves invariant with respect to the isogonal transformation of $B_0B_1B_2B_3$, the isogonal pairs on them must form an involution, the double points of which can only be at A_0, that is to say, at the points at infinity on the rectangular

* Extremities of diameters of rectangular hyperbolas through B_0 B_1 B_2 B_3 are generally called 'twin points' with respect to the triangle B_1 B_2 B_3; the name is, from our point of view, open to the objection that it does not indicate that the transformation belongs to the same general type as the isogonal transformation of a triangle.

hyperbola. Hence the isogonal pairs must be at the extremities of diameters of the rectangular hyperbola.

It may be mentioned that the theorem (5.7) furnishes a tetra-cyclic proof of the theorem, that all conics through an orthocentric tetrad are rectangular hyperbolas; for, it shews that the isotropics through A_0 are harmonic to the nodal tangents to any cyclic through $B_0B_1B_2B_3$ with a node at A_0.

The Theorem (8.6) is only one of the four theorems of similar character, the following being typical of the other three:

(8.7) All rational bicircular quartics which pass through $B_0 B_1 B_2 B_3$ and have their node at A_r $(r = 1, 2, 3)$ are orthogonal curves (that is, have perpendicular nodal tangents); any such quartic is cut by the circles of the co-axal system determined by its nodal osculating circles, in isogonal pairs of the tetrad $B_0B_1B_2B_3$.

An analogous theorem will hold for the isogonal transformation with respect to $(A_0 = \infty, A_1, A_2, A_3)$; it would be as follows:

(8.8) Every circular cubic which passes through $A_1 A_2 A_3$ and has a node at B_r, has perpendicular nodal tangents, and is invariant for the isogonal transformation of the triangle $A_1A_2A_3$; any such cubic is cut by the circles of the co-axal system of its nodal osculating circles in isogonal pairs of $A_1 A_2 A_3$ $(r = 0, 1, 2, 3)$.

Here and hereafter we use the phrase 'isogonal transformation of a triangle $A_1A_2A_3$' in the sense of 'the tetra-cyclic isogonal transformation defined by the tetrad $(A_0 = \infty, A_1, A_2, A_3)$'.

§ 9. The Tetra cyclic Geometry of the Triangle.

The idea, which has been suggested above, of developing the geometry of a triangle $A_1A_2A_3$, as the set of inversion properties concerned symmetrically with the tetrad $(A_0 = \infty, A_1, A_2, A_3)$, and in particular, with the isogonal, co-isogonal, and ortho-transformations

determined by it, furnishes the basis for a new synthesis of the properties of the triangle.

The subject of triangle-geometry is conspicuous as much by the richness of its results, as by the diversity of the methods which have helped to build it up; Elementary Geometry, Synthetic Projective Geometry, Invariant Geometry, Inversion Geometry, Affine Geometry, and other miscellaneous ideas have each made their own track in its territory. The idea which is offered here will evidently come under 'Inversion Geometry,' but will differentiate itself from the general inversion-geometry of the triangle, by a greater symmetry and unity of conception, and the possession of a well-defined method, accompanied, no doubt, by a certain limitation of scope. It is desirable to give a name to the group of geometrical relations which fall under this idea. Accordingly, *we shall mean by 'tetra-cyclic geometry of the triangle', the class of properties, which, when stated in inversional form, involve symmetrically the tetrad composed of the point at infinity and the vertices of the triangle.* It is clear that the isogonal, co-isogonal, and ortho-transformations of the triangle will play a fundamental role in this subject.

Since A_0 is the point at infinity, B_0, B_1, B_2, B_3 are the in-centres of the triangle $A_1A_2A_3$ (8.4). Also, the circles $C_0C_1C_2C_3$ should invert A_0 into B_0, B_1, B_2, B_3 respectively. Hence C_0, C_1, C_2, C_3 are the four mutually orthogonal circles whose centres are B_0, B_1, B_2, B_3. The twelve desmic circles become, for $A_0 = \infty$, the six sides of the quadrangle $B_0B_1B_2B_3$, and the six circles described on these sides as diameters.

The rectangular hyperbolas through the four in-centres are a special class of invariant curves of the second type; they are, in fact, the irreducible nodal curves of the second type, with node at A_0 (5.7). We shall refer to them as the 'invariant hyperbolas' of the triangle $A_1A_2A_3$.

(9.1) *The twelve desmic circles are invariant,* qua *loci, in regard to the isogonal and co-isogonal transformations.**

* It is only rarely, as in this theorem, that circles occur as *loci*; usually in the present subject, they are space-elements.

For, consider the desmic circle $B_2B_3A_2A_3$; since it passes through A_2A_3, its isogonal transform, *qua locus*, will be another circle through A_2A_3. Since however, it contains the points B_2B_3 which are invariant for the isogonal transformation, it follows that it is its own isogonal transform. A similar proof holds for the other cases.

Since A_0 is at infinity, it follows from (8.3), that

(9.2) *The centres of isogonal circles are isogonal conjugates; further two isogonal circles invert each vertex of the triangle* $A_1A_2A_3$ *into isogonal conjugates.*

We may use this theorem to obtain a relation between the radii of isogonal circles. Let p, p' be isogonal conjugates, d, d' their distances from the circum-centre, and R, the circum-radius of $A_1A_2A_3$. If r, r' are the radii of isogonal circles with centres at p, p', it is clear that there must subsist a homographic relation between r^2, r'^2, which is satisfied by $r^2 = r'^2 = 0$. Thus the relation must be of the shape $\frac{A}{r^2} + \frac{B}{r'^2} = 1$; further, the value $r^2 = \infty$ gives the point-circle at infinity, whose isogonal circle must be orthogonal to the circum-circle. Hence $r^2 = \infty$ corresponds to $r'^2 = d'^2 - R^2$. Hence the sought relation is

$$\frac{d^2 - R^2}{r^2} + \frac{d'^2 - R^2}{r'^2} = 1 \tag{9.3}$$

since each of the circles C_0, C_1 C_2, C_3, inverts the two associated tetrads into each other, we have:

(9.4) *A pair of isogonal circles is inverted by any of the circles* C_0, C_1, C_2, C_3, *into a pair of co-isogonal circles.*

The lines of the cubic complex Γ correspond to pencils of circles determined by a pair of isogonal circles. Such pencils may be called Γ-pencils. It follows that any Γ-pencil contains a pair of isogonal circles, a pair of co-isogonal circles, and a pair of ortho-circles, every two of these pairs being harmonic. But ortho-circles are necessarily orthogonal (8.1). Hence each of the ortho-circles inverts the two isogonal circles

into each other, and also the two co-isogonal circles into each other. In other words

(9.5) *Any two ortho-circles are the circles of anti-similitude of the pair of isogonal circles, as well as, of the pair of co-isogonal circles contained in their pencil.*

The circles of a Γ-pencil are in homographic correspondence with their centres. Hence the centres of the two ortho-circles, the two isogonal circles, and of the two co-isogonal circles in the pencil must form three mutually harmonic pairs. But the centres of the isogonal circles are isogonal conjugates (9.2),, and it is known that isogonal conjugates are conjugate points in regard to the pencil of invariant hyperbolas.* Hence the centres of the two ortho-circles, as also of the two co-isogonal circles, are cut out on the line of centres by invariant hyperbolas. Hence

(9.6) *The centres of two co-isogonal circles lie on the same invariant hyperbola; the centres of two ortho-circles also lie on the same invariant hyperbola.*

Two orthogonal pencils of circles are so related that the common points of either are the limit points of the other; from the Cartesian point of view, the two pairs of limit points of two orthogonal pencils of circles, and the pair of circular points form the three pairs of opposite vertices of the same complete quadrilateral. From the inversion point of view, the two pairs are point-circles obtained by pairing differently the same four isotropics. Two such point-pairs may be called *the cross pairs* of each other.

A pencil of circles may be called an *isogonal pencil* or *a co-isogonal pencil*, if its limit points are respectively isogonal or co-isogonal points. Such pencils correspond to lines of the fundamental congruences Q_{31}, Q_{32}. Since each of these congruences is its own polar reciprocal congruence in regard to Q_3, it follows that the orthogonal pencil of an isogonal or

* This fundamental property of the Cartesian isogonal transformation has not been proved here so far. Its inversion-proof will be found in the next section.

co-isogonal pencil, is also an isogonal or co-isogonal pencil, respectively. In other words:

(9.7) *If two circles intersect in a pair of isogonal (co-isogonal) points, their limit-points are also isogonal (co-isogonal) points.*

(9.8) Since each of the fundamental congruences is of order-class (3,3), a circle C contains, in general, three isogonal (co-isogonal) pairs; further, there exist, also three inverse pairs of C which are isogonal (co-isogonal) pairs. It is evident that the three latter pairs are the cross-pairs of the three former.

Theorem (6.2) now gives:

(9.9) *The three co-isogonal pairs contained in a circle C are the common harmonic pairs of every two of the three isogonal pairs contained iu it.*

A corresponding theorem would of course hold for the cross-pairs. The theorem (6.4) translates into:

(9.10) *The three pencils determined by every two of the three circles which are orthogonal to C and cut it in its three isogonal (co-isogonal) pairs, are co-isogonal (isogonal) pencils.*

The theorem (6.5) brings to light a remarkable relation between isogonal (or co-isogonal) circles. We may infer from it the following construction for the isogonal circle of a given circle C:

(9.11) *Draw the three circles C', C'', C''' which are orthogonal to C, and cut it in its three co-isogonal pairs. Any one of these three circles contains* three *isogonal pairs, out of which we select that one which is* not *harmonic to the co-isogonal pair situated on C (compare (9.9)). The three isogonal pairs thus selected lie on a circle, which is precisely the isogonal circle of C.*

By interchanging 'isogonal' and 'co-isogonal,' we obtain a corresponding

construction for the co-isogonal circle. An alternative statement of the property expressed in this construction is as follows:

(9.12) *Two isogonal (co-isogonal) circles* C, C′ *are so related that each of the three isogonal pairs on the former is concyclic with one of the three co-isogonal pairs on the latter: the three new circles thus arising, are all orthogonal to* C′ (C).

A set of four circles may be said to form an *isogonal* (*co-isogonal*) *quadruple*, when the pencils determined by every two of the circles are isogonal (co-isogonal) pencils. Such quadruples correspond to the congruence-tetrahedra studied in § 6. It follows that every circle is in general, a member of only one isogonal quadruple, and of only one co-isogonal quadruple. The following property corresponding to (6.6), is characteristic of these quadruples:

(9.13) *The isogonal (co-isogonal) circle of each of the circles of an isogonal (co-isogonal) quadruple is the common orthogonal circle of the other three; the four such isogonal (co-isogonal) circles therefore form a second isogonal (co-isogonal) quadruple. Further, two circles of an isogonal (co-isogonal) quadruple are inverted into each other by the co-isogonal (isogonal) circles in their pencil (6.11).*

A circle which is orthogonal to its isogonal circle, must also be orthogonal to its co-isogonal circle. The aggregate of such circles constitutes a quartic complex ϕ, which corresponds to the quartic surface, ϕ_4 previously studied. The isogonal or co-isogonal circle of a ϕ-circle is also a ϕ-circle. The complex ϕ may also be defined as the aggregate of co-isogonal circles contained in isogonal pencils, or as the aggregate of isogonal circles contained in co-isogonal pencils. Two of the four ϕ-circles contained in an isogonal (co-isogonal) pencil are thus the two co-isogonal (isogonal) circles contained in the pencil; the two remaining circles may be called *the residual ϕ-circles* of the pencil. From (6.10), we have the property of these residual ϕ-circles.

(9.14) *The isogonal (co-isogonal) circles of the residual ϕ circles in an isogonal (co-isogonal) pencil, are the residual ϕ-circles of the orthogonal pencil.*

Since the centres of isogonal circles are isogonal conjugates (9.2), this gives a simple construction for finding the residual ϕ-circles in an isogonal pencil. We have clearly the theorem :

(9.15) *The centres of the residual ϕ-circles in an isogonal pencil whose point-circles are at the isogonal conjugates p, q are the two points in pq whose isogonal conjugates lie in the perpendicular bisector of pq.*

By (6.12) these residual circles are inverted into one another by the two ortho-circles in the pencil. If we denote then, the centres of the two residual ϕ-circles in (9.15) by ϕ_1, ϕ_2, it follows that the two points which separate $\phi_1\phi_2$ and pq harmonically, are the centres of the two ortho-circles in the isogonal pencil. Hence also can be determined the centres of the two co-isogonal circles in the pencil.

By (6.14), the focal circles and the two residual circles form a syzygetic tetrad. Hence:

(9.16) *The two focal circles in an isogonal (or co-isogonal) pencil are respectively orthogonal to the two residual ϕ-circles in the pencil. Therefore, their centres are the harmonic conjugates with respect to the limit-points of the centres of the ϕ-circles. Also, the two co-isogonal (isogonal) circles in an isogonal (co-isogonal) pencil invert the residual ϕ-circles into the two focal circles in the pencil.*

The construction of (9.15) has to be modified for the case of co-isogonal pencils. If pq are co-isogonal points, it will be a question of finding the two points $\phi_1\phi_2$ on pq, which are the centres of those circles of the pencils, whose co-isogonal circles belong to the orthogonal pencil. It is best for this purpose to represent the circles parametrically, not by their centres, but by the points into which they invert one of the points B.

The circles, on which two of the contained isogonal (and therefore also co-isogonal) pairs coalesce may be called *coincidence circles*. Such circles constitute the octavic *coincidence complex* which corresponds to F_8 An isogonal or co-isogonal pencil contains eight coincidence-

circles, of which four coalesce in pairs at the two focal circles of the pencil, and the remaining four form two pairs each of which is inverted into itself by the two ortho-circles in the pencil.

The theorem (9.15) may be used for the construction of the ortho-circle of a given circle C. A more symmetrical construction may however be derived by using (6.16); this shews that:

(9.17) *The ortho-circle of C is that orthogonal circle of C, whose centre is the centre of perspective of the two polar triangles, whose sides cut out respectively the isogonal and co-isogonal pairs on C; it follows also that the axis of perspective of these triangles is the radical axis of C and its ortho-circle.*

Isogonal, co-isogonal, and ortho-transforms of special circles.

(*a*) *Line-circles.*

Lines are circles through A_0, the point at infinity. The three isogonal pairs contained in a line are (1) the usual isogonal pair pp' contained in the line, according to the Cartesian theory; (2) the pairs A_0q, A_0q' where qq' are the intersections of the line with the circum-circle of $A_1A_2A_3$. The three co-isogonal pairs, being the common harmonic pairs of every two of these (9.9), are (1) the pair A_0A_0 (2) two pairs p_1p_2, $p'_1p'_2$ both harmonic to pp', and having their middle points at q,q' respectively. By (8.6), co-isogonal pairs of $A_1A_2A_3$ are extremities of a diameter of a hyperbola through its in-centres $B_0B_1B_2B_3$. Remembering that the circumcircle of $A_1A_2A_3$ is the centre-locus of these invariant hyperbolas, we see that the two co-isogonal pairs p_1p_2, $p'_1p'_2$ are cut out of the line by the invariant hyperbolas whose centres are q, q' respectively.

To find the isogonal circle of a line, we may either utilise the formula (9.3), and perform a limiting process, or use a direct method. Consider a family of parallel straight lines, represented parametrically by their perpendicular distance δ from the circum-centre. These may be considered as a family of concentric circles whose common centre is the infinite point A_0, approached in a perpendicular direction. If the

point a_0 on the circum-circle be the isogonal conjugate of this perpendicular direction, it follows from (9.2), that the isogonal circles of the lines, all have their centre at a_0. Hence r'^2, the square of the radius of the isogonal circle, must be a homographic function of δ. When $\delta = \infty$, the line becomes the point-circle A_0, and therefore the isogonal circle must be orthogonal to the circum-circle, so that $r'^2 = 0$. Again when $\delta = 0$, the line is orthogonal to the circum-circle, and so the isogonal circle is A_0, so that $r'^2 = \infty$. *Hence r'^2 is of the form* $\frac{K}{\delta}$. To find K, take $\delta = + R$, so that the line touches the circum-circle at p (say); let the invariant hyperbola whose centre is p cut the line in $p_1 p_2$, and write $pp_1 = pp_2 = r$. Let λ, λ' be the finite pair of isogonal conjugates on $a_0 p$; then, by (9.11), λ, λ' are the limit points of the circle on $p_1 p_2$ as diameter, and the isogonal circle of the line. Hence $r^2 = p\lambda . p\lambda'$ necessarily, and $r'^2 = a_0\lambda . a_0\lambda'$. Thus $K = R . a_0\lambda . a_0\lambda'$. Similarly $K = -R . a_0\mu . a_0\mu'$, where μ, μ' are the isogonal conjugates in the join of a_0 with the *other* possible point of contact corresponding to $\delta = -R$. We gather that $a_0\lambda . a_0\lambda' = -a_0\mu . a_0\mu'$.

It is known that the four quadratics, which represent parametrically the intersections of a given conic with the sides of a triangle, and with the axis of perspective of the triangle and its polar triangle, are such that their squares are linearly related. Using this fact and (9.17), we may easily find the ortho-circle of a line. Let the points on the line be represented by their distances from the middle point of the isogonal pair, so that $(t, -t)$ is the isogonal pair, and (r_1, r_2) the intersections of the line with the circum-circle. We can have an identical relation of the form:

$$r_2 (x - r_1)^2 - r_1 (x - r_2)^2 = \mu \{ (x^2 - \lambda^2)^2 - (x^2 - t^2)^2 \},$$

provided that

$$r_2 - r_1 = 2\mu (t^2 - \lambda^2)$$

$$r_1 r_2 (r_1 - r_2) = \mu \{ \lambda^4 - t^4 \};$$

that is, eliminating μ, if

$$2r_1 r_2 - t^2 = \lambda^2.$$

Thus the ortho-circle cuts the line in the points, $\lambda, -\lambda$ given by this equation, since the ortho-circle is orthogonal to the line (8.1), we have;

(9.18) *If O is the mid-point of the isogonal conjugates P, P′ on a line, and if OT is the tangent to the circum-circle, the orthocircle of the line is the circle with O as centre, and radius equal to* $\sqrt{2OT^2 - OP^2}$.

The centre of the co-isogonal circle of a line must lie in the invariant hyperbola which has an asymptote parallel to the line (9.6). This hyperbola is that which passes through the mid-point O of the isogonal pair in the line. Further, the isogonal, co-isogonal, and ortho-circles of any circle belong to a pencil, and therefore have collinear centres. Hence the centre of the co-isogonal circle of the line is the other intersection of this hyperbola with the join of O with the point a_0 on the circum-circle, which is the isogonal conjugate of the perpendicular direction.

(*b*) *Point-Circles.*

Let P_1 be the isogonal, and P_2 the co-isogonal conjugate of the point P, so that PP_2 is the diameter through P of the invariant hyperbola $B_0B_1B_2B_3$ P. The ortho-circle of P being orthogonal to P must pass through it. Since the isogonal, co-isogonal and ortho-circles belong to a pencil, it follows that:

(9.19.) *The ortho-circle of the point P, is that circle through P which inverts* P_1 *into* P_2; *it is therefore orthogonal also to the circle* PP_1P_2, *and has its centre at the intersection of the tangent at P to this circle, with* P_1P_2.

(*c*) *Other Special Circles.*

(9.20). The isogonal circle of the circum-circle is the polar circle. For, since the isogonal conjugate of the circum-centre is the ortho-centre H, it follows that the isogonal circle has its centre at H. Further, since the circum-circle inverts each of $A_1A_2A_3$ into itself, it follows from (8·3), that the isogonal circle must invert each of them into points on the opposite sides. This proves the theorem.

Let the radii of the four mutually orthogonal circles $C_0C_1C_2C_3$ whose centres are $B_0B_1B_2B_3$ be $R_0R_1R_2R_3$ respectively. Since B_0 is its own isogonal conjugate, the system of circles whose centre is B_0, is composed of pairs of isogonal circles; in particular, C_0 is its own isogonal circle. Hence if R, R′ be the radii of isogonal circles with centre B_0, we must have

$$\frac{1}{R^2}+\frac{1}{R'^2}=\text{constant}=\frac{2}{R_0{}^2}.$$

The three isogonal pairs on the circum-circle are A_2A_3, A_3A_1, A_1A_2, and the three co-isogonal pairs, A_1A_1, A_2A_2, A_3A_3. The centre of perspective of the corresponding triangles is the Lemoine point, which is therefore the centre of the ortho-circle of the circum-circle. If O, K, H are respectively the circum-centre, Lemoine point, and ortho-centre, the centre of the co-isogonal circle of the circum-circle is the other intersection of HK with the invariant hyperbola through OK.

We may construct the isogonal quadruple of which the circum-circle is one member. The three other circles of the quadruple must all be orthogonal to the isogonal circle of the circum-circle, namely, the polar circle (9.20). Now, the three isogonal pencils containing the circum-circle are those whose common points are (A_2A_3), (A_1A_2). Let the perpendicular bisector of A_2A_3 cut the circum-circle in $\delta_1\delta_2$; then $\delta_1\delta_2$ are the two points in the bisector, whose isogonal conjugates lie in A_2A_3. Hence, by (9.15), δ_1, δ_2 are the centres of the residual ϕ-circles in the pencil through A_2A_3. Hence, if D_1 is the mid. point of A_2A_3, the centres of the co-isogonal circles in the pencil are a pair of x, y in $\delta_1\delta_2$, which are conjugate in regard to the circum-circle, and are such that $D_1x \,.\, D_1y = - D_1A_2{}^2$. The second circle of the isogonal quadruple in the pencil through A_2A_3 is the inverse of the circum-circle in either of the co-isogonal circles (9.13). It is also the circle orthogonal to the polar circle. Hence, we see that *the three circles which form an isogonal quadruple with the circum-circle are the three circles, orthogonal to the polar circle, whose radical axes with the circum-circle are the sides of the triangle.*

The isogonal quadruple containing a line-circle may be similarly specified.

§ 10. The Three Invariant Types of Cyclic Curves.

At the close of section 5, we studied the three types of invariant quartic curves on the quadric Q_3, cut out by the three systems of fundamental quadrics. The first and second types of invariant curves are cut out by quadric of the second and first system, and are therefore circumscribed to Δ_1, Δ_2, and are invariant for Γ_1, Γ_2 respectively. The invariant curves of the third type are those cut out by other quadrics of the third system, and are invariant for Γ_1 as well as Γ_2. The first two types are linear ∞^2-systems of curves, while the third type is a *pencil*, or linear ∞^1-system ; no invariant curve of the first or second type is a fundamental curve, whereas the invariant curves of the third type are all the fundamental curves on Q_3. Finally, recalling a previous result (5.7), the only irreducible nodal invariant curves of the first or second type, are those whose node is situated at a vertex of Δ_2 or Δ_1, respectively, and possess the property of being orthogonal, that is, of having perpendicular nodal tangents.

In studying these curves in plane geometry, it will be convenient to use Cartesian trilinear co-ordinates with respect to the triangle $A_1A_2A_3$. The in-centres $B_0B_1B_2B_3$ have then the co-ordinates $(\pm 1, \pm 1, \pm 1)$, and 'the circular points,' the co-ordinates, $(e^{-iA_2}, e^{iA_1}, -1)$. The equation to the isogonal transformation in these co-ordinates, is:

$$\Gamma_1: \qquad x_1' : x_2 : x'_3 = x_2x_3 : x_3x_1 : x_1x_2.$$

The rectangular hyperbolas through the in-centres are of the form $\lambda x_1^2 + \mu x_2^2 + \nu x_3^2 = 0$, $\lambda + \mu + \nu = 0$. By finding the co-ordinates of the extremities of a diameter of such a hyperbola, we may shew that the cartesian co-isogonal transformation is given by the equations:

$$(10.1.) \qquad \Gamma_2 :: \qquad x_1' : x_2' : x'_3 =$$

$$2a_1 \Sigma a_1 x_1 (x_3^2 - x_1^2)(x_1^2 - x_2^2) - x_1 \Sigma a_1^2 (x_3^2 - x_1^2)(x_1^2 - x_2^2)$$
$$: 2a_2 \Sigma a_1 x_1 . (x_2^2 - x_2^2)(x_2^2 - x_3^2) - x_2 \Sigma a_1^2 (x_3^2 - x_1^2)(x_1^2 - x_2^2)$$
$$2a_3 \Sigma a_1 x_1 . (x_2^2 - x_3^2)(x_3^2 - x_1^2) - x_3 \Sigma a_1^2 (x_3^2 - x_1^2)(x_1^2 - x_2^2).$$

Here a_1, a_2, a_3 are the sides of the triangle $A_1A_2A_3$. The equation shews that from the Cartesian point of view, the co-isogonal transformation is quintic Cremona involution, with six fundamental points, all of the second order, situated at the in-centres and the circular points; thus a line transforms into a rational bicircular quintic, which has finite double points at the in-centres, and has its asymptote parallel to the line. *

The first type of invariant cyclic curves evidently becomes in the plane, the family of circumscribed circular cubics, which are invariant for the isogonal transformation, and do *not* pass through the in-centres. The equation to this family is seen to be:

$$(10\ 2)\ \mathrm{I}\quad \begin{aligned} & l x_1(x_2^2+x_3^2) + m x_2(x_3^2+x_1^2) + n x_3(x_1^2+x_2^2) + p x_1 x_2 x_3 = 0, \\ & \text{where}\quad 2l \cos A_1 + 2m \cos A_2 + 2n \cos A_3 + p = 0. \end{aligned}$$

The curves of this type which pass through an in-centre (1, 1, 1, 1), satisfy the further relation $2(l+m+n)+p=0$; they are easily verified to have a node at the incentre, with perpendicular nodal tangents.

The invariant curves of the second type are bi-circular quartics which are invariant for the co-isogonal transformation, and pass through the in-centres, but not through the vertices of the triangle. They may be obtained by inverting the invariant curves of the first type in one of the circles $C_0 C_1 C_2 C_3$. It will be noticed also that the rectangular hyperbolas through the in-centres belong to this family.

The invariant curves of the third type are the circular cubics which pass through the in-centres and are circumscribed to the triangle. They are of the form;

$$(10.2)\ \mathrm{III.}\quad \begin{aligned} & l x_1(x_2^2 - x_3^2) + m x_2(x_3^2 - x_1^2) + n x_3(x_1^2 - x_2^2) = 0; \\ & \text{where}\quad l a_1 + m a_2 + n a_3 = 0. \end{aligned}$$

The locus of isogonal pairs on lines through a given point P is a self-isogonal cubic (which has been called 'the Jumelaire cubic of P †) of the

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* This Cremona Transformation has been investigated by P. Schoute, *L' Intermediare des Math.* 5. (1898). See also *Ency. Math. Wiss III.* 5.

† Malet: *Transformations Birationelles*, Ganthier-Villars.

form III, possessing the property that its tangents at the in-centres meet at P. This cubic becomes a circular cubic, and therefore an invariant curve of the third type, when P is at infinity. Hence:

(10.3) *The invariant curves of the third type are the loci of isogonal pairs contained in a family of parallel lines; similarly they are the loci of isogonal pairs contained in a pencil of circles, which have contact at a vertex of the triangle $A_1A_2A_3$. From symmetry, they are also the loci of co-isogonal pairs contained in a pencil of circles which have contact at any one of the in-centres.*

This theorem might have also been deduced from the geometry of the transformations Γ_1, Γ_2 on the quadric Q_3. The theorem may also be stated as follows:

(10.4) An invariant curve of the third type is cut by any circle which touches it at A_1, A_2 or A_3, or by any line parallel to its asymptote, in a pair of isogonal points: it is also cut by any circle which touches it at one of the in-centres in a pair of co-isogonal points.

Several properties follow from this theorem. For instance, the tangent to the curve at each vertex meets the curve again at the same point λ on the circum-circle, and this point also lies on the asymptote; the parallels to the asymptote through each vertex meet the curve on the opposite side. The circles of curvature at the vertices also pass through these meets; by the same reasoning the circle of curvature at infinity (namely, A_0), that is to say, the *asymptote*, passes through the point λ. Analogous results hold for the in-centres.

(10.5) *The joins of co-isogonal pairs on an invariant curve of the third type pass through the fixed point λ.*

This will become clear if we remember that there are only two kinds of birational involutions on a cubic curve; one of these consists of pairs of points collinear with a fixed point on the curve and possesses four

double points, while the other kind consists of pairs of points, the tangents at which meet on the curve, and does *not* possess any double points. The involution of co-isogonal pairs on the invariant curve of the third type evidently belongs to the former category, and the theorem follows.* It follows from this that we can also define an invariant curve of the third type as the locus of co-isogonal pairs on lines through a fixed point λ on the circum-circle. It however remains to be examined, which of the three co-isogonal pairs on the line traces the curve.

Since an invariant curve of the first type Σ_1 does not pass through any of the in-centres, it is clear on the other hand that the involution of isogonal pairs upon it, must be of the second category; that is to say, the tangents at isogonal conjugates on Σ_1 meet on the curve. From any point of Σ_1 four tangents can be drawn to the curve; their four points of contact can be divided into two pairs in *three* ways. The pairs obtained in only *one* of these ways will be isogonal conjugates. If, as we have been doing, we view this theorem from stand-point of inversion-geometry, we have to say that the circles through A_0, which touch Σ_1 at isogonal points meet again on the curve. Since the invariant curve is symmetrically related to $A_0A_1A_2A_3$, we have:

(10.6) **The tangents to Σ_1 at isogonal points meet on the curve; the circles through a vertex of the triangle which touch Σ_1 at isogonal points also meet again on the curve.**

The co-isogonal involution on an invariant bicircular quartic Σ_2 of the

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* Here the double points on the curve of the co-isogonal involution, appear from the Cartesian point of view, to be $A_1A_2A_3$ and the third point of intersection with the line at infinity. It is important to state in this connection that every point on the line at infinity is a fixed point of the Cartesian co-isogonal transformation; this corresponds to the tetracyclic fact that A_0 is a double point of Γ_2 *in every direction of approach.* The relation of the invariant curves of the third type to the isogonal and co-isogonal transformations is, in fact, unsymmetric, from the Cartesian view-point; it is however interesting to observe how the unsymmetry so adjusts itself as to be reconcilable to the original symmetry of the tetracyclic configuration.

second type is also of the same category. From the usual theory of the bicircular quartic, it follows therefore that there exist generating circles of different systems of Σ_2, which have *one* of their contacts identical, and their other contacts at two co-isogonal points on Σ_2. Analogously to (10.6), we would also have the theorem:

(10.6) The circles which pass through an in-centre, and touch Σ_2 at co-isogonal points, meet again on the curve.

Consider now the intersections of Σ_1 and an invariant curve Σ_3, of the third type. Since both the curves are invariant for the isogonal transformation, the intersections form two isogonal pairs; from the property of Σ_3 (10.3), it follows that these pairs are situated on parallel lines. Hence any curve Σ_1 contains two isogonal pairs whose joins have a given direction, and these are cut out of it by an invariant curve of the third type.

Now any curve Σ_3 is completely determined by *one* isogonal pair. It follows therefore that the pencil of curves Σ_3 determine an involution between the isogonal pairs contained in a given Σ_1. Now the totality of isogonal pairs on Σ_1 is an algebraic domain of genus unity, since the envelope of their joins is an elliptic curve of class three, touching the line at infinity; the involution determined on this algebraic domain by the curves Σ_3 must be of the first category, since the joins of corresponding isogonal pairs are *parallel*, that is to say, meet on a fixed tangent of the envelope. Hence:

(10.7) Any curve Σ_1 has double contact with four curves Σ_3, the chords of contact being the four asymptotes of the envelope of joins of isogonal pairs on Σ_1.

A similar result may be stated for the invariant curve of the second type. It is also easy to see that the isogonal conjugate of the tangential of an isogonal pair is the third intersection of its join with Σ_1.

In particular, the double focus of Σ_1 lies on the curve, and must be the fourth intersection of the circum-circle with it, since it is the isogonal conjugate of the third point at infinity on Σ_1.

(10.8) If PP', QQ' are two isogonal pairs on Σ_1, such that PP' is parallel to QQ', then PQ, $P'Q'$ intersect Σ_1 at the same point R, and PQ', $P'Q$ intersect it at the same point R', and R and R' are isogonal conjugates, and constitute, in fact, the two points of contact of the remaining tangents from the double focus f. Further the other intersections of PP', QQ' with the curve are collinear with f.

It is clear that a theorem like this should hold; for the isogonal pairs PP' on Σ_1 are in (2, 1) correspondence with the third intersections of PP' with Σ_1. Since the involution of isogonal pairs PP', QQ' is of the first category, the corresponding third intersections should be collinear with a fixed point f on Σ_1. To shew that f is the double focus, represent the points of Σ_1 by an elliptic parameter ω, with $\omega_1+\omega_2+\omega_3=0$ (mod. 2ω, $2\omega'$) as the condition of collinearity; the isogonal conjugates PP' will then have their parameters related by the equation $P'=P+\omega$ Hence

$$R = -(P+Q) = -(P'+Q')$$

$$R' = -(P+Q') = -(P'+Q) = -(P+Q)+\omega$$

$$f = P+P'+Q+Q' = 2(P+Q).$$

This shews that f is the tangential of R and R', so that R, R' are fixed points on the curve possessing the property, that, if PP' be any isogonal pair on Σ_1 and QQ' the second isogonal pair defined by $Q=-(P+R)$, then PP' and QQ' are parallel. Taking P, P' at RR', we see that RR' is parallel to the asymptote. Since f is the isogonal conjugate of the third intersection of RR' with the curve, it must be the double focus.

To interpret geometrically the co-isogonal involution on Σ_2 an invariant curve of the second type, and also the involution of co-isogonal pairs cut out on Σ_2 by curves of the third system, we may use the same method of parametric representation, with $u_1+u_2+u_3+u_4=0$ (mod. 2ω, $2\omega'$) as the condition of concyclic position. The co-isogonal involution on the curve is, as before, of the type $u'=u+\omega$. The curve contains pairs of tetrads of the form $(u, u+\omega, u+\omega+\omega', u+\omega')$, $(-u, -u+\omega, -u+\omega', -u+\omega+\omega')$, so related that any point of the

first and any point of the second are contacts of a generating circle. Every such tetrad is composed of two associated co-isogonal pairs. These tetrads form an irrational involution (D′) of genus *one*, on the curve. Accordingly they can be put in birational correspondence with points x on the curve; we may do this by choosing a fixed point a on the curve, and taking for x the other common point of the four circles which pass through a, and touch the curve at points of the tetrad. If two tetrads are related by an involution of the first category in the domain D′, then the corresponding points x are concyclic with a and a fixed point b of the curve.

The co-isogonal pairs on Σ_2 also form a domain D, of genus *one*; the involution of associated co-isogonal pairs, which generates the domain D′, is obviously an involution of the second category in D. The two co-isogonal pairs cut out on Σ_2 by an invariant curve of the third type, are connected by an involution of the first category. Now, as may be easily verified, two involutions of different categories in the same elliptic domain are necessarily permutable, their product being an involution of the first category. Thus the two tetrads to which the two co-isogonal pairs cut out by a Σ_3 pertain, are related in an involution of the first category in D′. Hence, if PP′, QQ′ be the two isogonal pairs cut out by a Σ_3, and if the circles aPP, bQQ cut the curve again in x, x', then the circle axx' passes through a fixed fourth point on the curve. This result may also be obtained by inverting the corresponding result for curves of the first type.

It is also important to notice that each of the three systems of invariant curves contains six pairs of circles, which are different pairings of the same twelve desmic circles, namely, the six sides of the quadrangle of in-centres, and the six circles described on them as diameters. For instance, the pairs B_0B_1 and the circle on $B_2 B_3$ as diameter constitute an invariant curve of the third system; B_0 B_1 with the circle described on it as diameter, forms an invariant curve of the first type; B_0B_1, B_2B_3 together constitute an invariant curve of the second type, as also the circles described on them as diameters. Hence, also, the desmic circles are invariant *qua* loci, for the isogonal and co-isogonal transform-

ations, as has been already proved. Several special properties of the invariant curves in reference to the desmic circles may also be developed.

§ 11. The Polar and the Apolar Theory of the Invariant Cyclic Curves.

The Apolar Theory of the Cyclic.

The inversion-polar theory of the cyclic, and its relation to the quaternary polar theory of the quadric, stated in 1900 by E. Kasner* have not yet been utilised as a method in the research of these curves. We shall obtain here the inversion-form of the Cartesian theorem that isogonal points are conjugate in respect of all hyperbolas through the incentres, and shall incidentally make some contributions to the apolar theory of the general cyclic. We shall have to deal also with the question of the form, which the apolar theory of the general cyclic takes, for the conic-cyclic.

The plane cyclic can be considered as a (2, 2) curve on the quadric manifold Q_3 whose elements are point-circles. The following *á priori* development of the apolar theory of the (2, 2) curves on Q_3 justifies Kasner's Transformation-principle.

We take the form-theoretic definition of apolarity for starting point. Namely, we let the binary parameters x, y represent the generators of the two systems on Q_3, so that the (2, 2) curve corresponds to a (2, 2) form $a_x^2\, b_y^2$ in such a way, that the parameters x, y of the generators which meet on the curve are connected by $a_x^2\, b_y^2 = 0$. Two (2, 2) forms $a_x^2\, b_y^2$, $a'^2_x\, b'^2_y$ are *apolar* if $(aa')^2\, (bb')^2 = 0$; that is to say, apolarity is a symmetric lineo-linear relation, invariant for linear transformations of x and y. Now a circle corresponds to a bilinear form, and is represented by a plane section P of Q_3, or by its pole p. Hence two circles which are *conjugate* with regard to the cyclic $a_x^2\, b_y^2$ — that is—two circles which together make up a cyclic apolar to $a_x^2\, b_y^2$—, must therefore

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* E. Kasner: *Invariant Geometry of the Inversion Group.* Trans. Am. Math. Soc. 1900. The Ency. Math. Wiss. does not mention the polar theory in its article on Bicircular Quartics.

be connected by a symmetric bilinear relation. Hence also the corresponding points p, q must be connected by a similar relation, and therefore must be conjugate either in a Null System or in a Quadric Polarity. If the former were the case, then any squared circle would be identically apolar to $a_x^2 b_y^2$. This is not so; for instance, a squared point-circle whose co-ordinates are $(x_1 y_1)$ is apolar to $a_x^2 b_y^2$, when $(xx_1)^2 (yy_1)^2$ is apolar to it, that is, when $a_{x1}^2 b_{y1}^2 = 0$, that is to say, only when the point is on the curve. This shews that p, q must be conjugate in the polarity of a quadric Q which *contains* the cyclic curve. Q must be a rational co-variant quadric of Q_3 in the pencil determined by the cyclic, and therefore can only be its out-polar quadric. The polar plane P of p in regard to this out-polar quadric is thus the locus of points which correspond to circles conjugate to the circle p. If p' is the pole of P in regard to Q_3, then all these conjugate circles are orthogonal to the circle p'; naturally we call p' *the polar circle* of p in regard to the cyclic, and reciprocally, the circle p, *the anti-polar circle* of p'. The points p and p' are related in a collineation (which is the result of performing successively the polarities of Q, Q_3) whose fixed points are the vertices of the self-polar tetrahedron of the curve; the multipliers of the collineation will depend on the characteristic roots of the pencil of the matrices of Q, Q_3. The same argument shews the geometrical relation between *apolar cyclics*. For, a linear combination of a sufficient number of conjugate circle-pairs of $a_x^2 b_y^2$ should constitute a general cyclic apolar to $a_x^2 b_y^2$. Now the pair of points pp' is, tangentially, an inpolar quadric of Q; and the corresponding pair of conjugate circles p, p' is the contact-cyclic of Q_3 with the envelope-quadric pp'. Hence:

(11.1) If a quadric Q is out-polar to the quadrics Q_3 and Q', then the intersection-cyclic of Q_3 with Q is apolar to the contact-cyclic of Q_3 and Q'.

This is Kasner's transformation-principle stated for cyclics. The polar theory of the cyclic follows from this. It will be noticed that the apolar relation of cyclics is symmetric and self-dual unlike the apolar relation of quadrics.

As in Cartesian Geometry, the polar circle of any point on the cyclic

has *contact* with the cyclic at the point. But the Cartesian presumption that the polar in regard to a nodal curve passes through the node is not fulfilled. For, suppose the cyclic to have a node n; then the out-polar quadric Q is in general a non-singular quadric which touches Q_3 at n. Therefore, the polar plane of an arbitrary point in regard to Q need *not* pass through n, which proves our statement. If however Q is a cone with vertex n, then the polar plane must always pass through n, so that the polar circle of every point passes through the node. To examine what this implies in regard to the cyclic, we recall that a cone with an arbitrary vertex n is *out-polar* to Q_3, if its section by any plane is out-polar to the section of the tangent cone to Q_3 from n If n is on the quadric, the tangent cone, regarded as an envelope, is the pair of generators through n. Hence, the cone, if it is to be out-polar to Q_3 must have the generators through n for conjugate lines. In other words, the nodal tangents of the cyclic must separate harmonically the nodal generators. If we suppose the node n to be the point at infinity of tetracyclic geometry, the nodal cyclic will be a conic, and on account of the harmonic property, a rectangular hyperbola. We have thus:

(11.2) *The ternary polar theory of the Cartesian conic Σ can agree with the double-binary polar theory of the nodal cyclic Σ, only when Σ is a rectangular hyperbola.*

It remains to verify that the polar theories actually agree for a rectangular hyperbola. From the harmonic property of pole and polar, it is easy to shew that the polar circle of a point p on Q_3 is the locus of points on circles through pn, which are harmonically conjugate to p in regard to the intersections with the cyclic. The mere fact that the polar circle does not pass through n, except when the nodal tangents are harmonic to the nodal generators, suffices to shew that this generation of the polar circle will not be valid for other cases. This agreement of the polar theories is of significance.

The agreement of (11.2) is, be it noted, only in regard to the polar circles of *point-circles*. Since polarisation with respect to the rectangular hyperbola, is equivalent to quaternary polarisation with respect to a

cone, whose vertex is the point corresponding to the point at infinity, we have further:

(11.3) *The polar circle of any circle with regard to the rectangular hyperbola is the polar line of its centre; thus, the anti-polar circle of a general circle is the point at infinity, while the antipolar circle of a line-circle has its centre at the pole of the line, but is otherwise indeterminate.*

A method for obtaining the polar circle of a given circle in regard to a general conic is given below.

A Construction for the Polar Circle.

We shall now prove a theorem on the polar circle, which bears an analogy to the Cartesian definition of the polar line as *the chord of contact of tangents*. Let γ be a general cyclic on Q_3 and p any point. The polar planes of p in regard to quadrics through γ will have a common line l. In particular, l will be contained in the polar planes of p in regard to each of the four cones containing γ. If C be one of these cones, then the corresponding polar plane is the plane of the lines of contact of the tangent planes from p to C. Now the tangent planes of C are bitangent planes of the cyclic; the sections of Q_3 by them, are accordingly generating circles of one system of the cyclic. Thus the section of Q_3 by the polar plane is *the circle through the contacts of the two generating circles of a particular system, which are orthogonal to the given circle p*. Let us mean by 'the contact-circle of two generating circles G_1, G_2 of the same system', the circle which passes through the four points of contact of G_1 G_2 with the cyclic. Accordingly, the four such contact-circles obtained from the four systems of generating circles, must belong to a co-axal system containing the given circle p, as well as its polar circle. The invariant relation of p and its polar circle in this system may be derived from the invariant relation of a pair of apolar quadrics in a pencil, to the four cones of the pencil. We have thus the theorem:

The coaxal system determined by a circle p and its polar circle in regard to a cyclic γ contains the four contact-circles of the four pairs of generating circles, which are orthogonal to p;
(11.4) The polar circle is, parametrically, the third polar of the circle p, in regard to this tetrad of contact-circles.

If the cyclic becomes a conic, it has two proper systems of double-contact circles; the other two systems coalesce into the tangent lines of the conic. The theorem may be used for finding the polar circle of a given circle in regard to a conic.

Doubly apolar quadrics.

Two general quadrics have a unique common self-polar tetrahedron. The relations between two quadrics are best studied through the ∞^3 system of quadrics having a common self-polar tetrahedron. To find the geometrical relation between two doubly-apolar quadrics, we may put the ∞^3—system of quadrics $a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2$ in correspondence with points A in a three-space, whose homogeneous co-ordinates in regard to a tetrahedron of reference Δ_1 are $(a_0a_1a_2a_3)$; the vertices of Δ will thereby correspond to the four squared planes of the system. The in-polar and out-polar quadrics of $\Sigma\, a_0x_0^2$ correspond to the polar cubic and the polar-plane of A in respect of Δ; these intersect in the three lines A_1A_2, A_2A_3, A_3A_1, where $AA_1A_2A_3$ is that desmic tetrahedron of Δ which has one vertex at A. Thus the doubly apolar quadrics oonstitute three pencils corresponding to these three lines. Now the polar reciprocal of $b_0x_0^2 + b_1x_1^2 + b_2x_2^2 + b_3x_3^2$ in regard to $a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2$ is $\Sigma \dfrac{a_r^2}{b_r} x_r^2$. Hence polar reciprocation is equivalent to a cubic transformation of the type we have been considering, with singular points at the vertices of Δ and fixed points at $AA_1A_2A_3$ and the vertices of the third tetrahedron of the desmic system. Hence the line A_1A_2 is transformed into itself by the reciprocation; remembering that A_1A_2 meets a pair of edges of Δ, it follows that ths quadrics corresponding to points on A_1A_2 have four common generators. Hence the property which characterises two doubly apolar quadrics Q, Q′

is that each has four lines in common with its polar reciprocal in regard to the other; in other words,

(11.5) Each of two doubly apolar quadrics must contain two mutually intersecting pairs of polar lines of the other; conversely two quadrics thus related are doubly apolar.

From this theorem, we may deduce an important property of the intersection-cyclic γ of Q_3 with a doubly-apolar quadric Q.

By this theorem Q contains a pair of polar lines of Q_3, which intersect γ in pq, $p'q'$ (say). From the property of polar lines it follows that pp', pq', qp', qq' are all generators of Q_3. Thus the (2,2) correspondence between generators of Q_3 which meet on γ, possesses two pairs (pp', qq'), $(pq', p'q)$, such that each generator of the former pair corresponds to each generator of the latter. This property of a (2,2) correspondence is poristic; the correspondence as well as the curve γ determining it, may then be called *subrational*. It is clear that the subrational cyclics γ, on Q_3, are those which admit one, and therefore ∞^1 inscribed generator-quadrangles. Conversely, it may be shown that cyclics with this property are necessarily cut out by doubly-apolar quadrics.* We have therefore the result.

(11.6) *The subrational cyclics on Q_3 are its intersection-cyclics, as well as its contact-cyclics with quadrics doubly apolar to it.*

If the subrational curve γ on Q_3 is cut out by Q, then, if lm, $l'm'$ are the pairs of polar lines of Q contained in Q_3, it is clear that any generator of Q_3 cuts γ and one of the pairs of lines lm, $l'm'$ harmonically. But this is the condition satisfied by conjugate cyclics. Hence the conjugate of the subrational cyclic γ is composed of the four isotropics lm, $l'm'$. This property may also be used to define subrational curves.

The H-cyclic.

The intersection-tetrad and the contact-tetrad of a base conic with a conic out-polar to it, are so related that the latter is the Hessian tetrad

* See my "*Algebraic (2,2) Correspondence,*" *Jour. Ind. Math. Soc.*, Feb. and Apr. 1926.

of the former. The Hessian tetrad belongs to the syzygetic family of a given tetrad, further each tetrad is the Hessian of *two* tetrads in the syzygetic family. The syzygetic family also contains two tetrads each of which is the Hessian of the other ; these are the two self-apolar tetrads in the family.

There are analogues in the theory of the cyclic, to these theorems from the invariant theory of the binary quartic. If γ is a cyclic curve on Q_3, the contact-cyclics of Q_3 with all quadrics through γ form the family of cyclics *confocal with* γ. Accordingly, *the confocal system* is the analogue of the syzygetic family of the binary theory.* The contact-cyclic with the out-polar quadric through γ (which will correspond to the Hessian tetrad) may be called the H-cyclic of γ. It is clear that any cyclic is the H-cyclic of three curves in its confocal system. The analogue of tetrads, each of which is the Hessian of the other is given by the following theorem :—

11.7) *A confocal system contains three pairs of sub-rational curves; the two curves of each pair are so related that each is the H-cyclic of the other.*

For, since the confocal system is contained in a co-director system, we may study it through the representation of quadrics of the form $\Sigma\, a_r x_r^2$ by the points whose co-ordinates in regard to a tetrahedron Δ are $(a_0 a_1 a_2 a_3)$. Let A be the point corresponding to Q_3, $AA_1A_2A_3$ the desmic tetrahedron of Δ with vertex A. The co-director cyclics on Q_3, correspond to pencils of quadrics of this family, which contain Q_3, and are hence represented by lines through A, and therefore also, by the points of intersection of these lines with the plane $A_1A_2A_3$, which is the locus of points representing quadrics out-polar to A. The locus corresponding to a *tangential* pencil of quadrics containing Q_3 is a cubic curve through A and the vertices of Δ; from this we see easily that the confocal system corresponds to a conic F in the plane $A_1A_2A_3$, which passes through the

* However the confocal system is a *quadratic* system unlike the syzygetic pencil of binary quartics. From another point of view, the co-director system of cyclics may be considered as the analogue of the syzygetic pencil.

points $\delta_1\delta_2\delta_3\delta_4$ into which A projects the vertices of Δ. Since the sub-rational cyclics are represented by the sides of $A_1A_2A_3$, it follows that the confocal system contains three sub-rational pairs. To prove the second part, suppose γ is a sub-rational curve cut out by the doubly apolar quadric Q. Then the contact-cyclic γ' of Q_3 with Q is the H-cyclic of γ, and is also sub-rational, since Q is doubly apolar to Q_3. For the same reason γ is also the H-cyclic of γ', thus showing that the three pairs of sub-rational curves have the relation stated. We may call γ, γ' the *twin sub-rational curves* of each other.

An important property of twin sub-rational curves will be obvious from the definition of polar circles in regard to a cyclic; namely,

(11.8) *If γ, γ' are twin sub-rational curves, and if C′ is the polar circle of C in respect of γ, then C is the polar circle of C′ in regard to γ'.*

We may also mention one further property of confocal systems, which it is quite easy to prove by the method used above:

(11.9) A confocal system contains an infinity of sets of three cyclics, every two of which are apolar.

It is also useful to mention that an arbitrary quartic curve γ is sub-rational on *six* quadrics through it; namely, the three pairs of doubly apolar quadrics contained in the pencil determined by γ.

A Criterion for the apolarity of two cyclics.

Let the quadric Q_3 cut an outpolar quadric k_1 in the cyclic γ_1; if its contact-cyclic with an inpolar quadric k_2 be γ_2, then by 11.1, γ_1, is apolar to γ_2 if k_1 is outpolar to k_2. But k_2 is already inpolar to Q_3. Hence, for apolarity, k_2 should be inpolar to every quadric through γ_1, and in particular to each of the four cones through γ_1. Hence each of these cones admit tetrads of generating lines, the plane of any two of which, is conjugate to the plane of the other two in regard to k_2. Now the tangent-planes of these cones correspond to generating circles of γ_1,

and the planes through twos of four generating lines correspond to contact-circles of twos of four generating circles. Hence:

(11.10) *If two cyclics γ_1, γ_2 are apolar, then there exist ∞^3 tetrads of generating circles of either, such that the contact-circle of any two circles of the tetrad is conjugate to the contact-circle of the other two, with respect to the other cyclic. Also there exist ∞^1 triads of generating-circles of either, such that the contact-circle of any two circles of a triad, is conjugate to every circle through the contact-points of the third, with respect to the other cyclic.*

This criterion for apolarity will hold if the cyclics γ_1, γ_2 are nodal, without having the same node. If, however, they have the same node p, then k_2 has contact with Q_3 at p, and the cone which projects from γ_1 from p counts twice among the four cones through p. The curve of intersection of k_2 and Q_3 is the H-cyclic of γ_2. Hence the condition of apolarity reduces to the harmonic separation of the nodal directions of γ_1 by the nodal directions of the H-cyclic of γ_2. Thus the criterion reduces to a binary invariant condition between the nodal directions of γ_1, γ_2 and the directions of the nodal generators. If p_t^2, q_t^2 represent the nodal directions of γ_1, γ_2, and r_t^2 the directions of the nodal generators, it may be shewn that the nodal directions of the H-cyclic of γ_2 are given by:

(11.11) $$2\,(rr')^2\, q_t^2 - 3\,(qr)^2\, r_t^2.$$

Hence the condition of apolarity of γ_1, γ_2 is:

(11.11) $$2\,(pq)^2\,(rr')^2 - 3\,(pr)^2\,(qr)^2 = 0.$$

It may be easily shewn that a nodal cyclic is sub-rational, only when it has orthogonal nodal tangents (that is, only when it is inversionally equivalent to a rectangular hyperbola). Thus when one of the curves γ_1, γ_2 — say, γ_2 —, is orthogonal, the above apolar condition reduces simply to

(11.11) $$(pq)^2 = 0$$

which expresses that the nodal directions of γ_1, γ_2 are mutually harmonic. Thus the conic-cyclics apolar to a rectangular hyperbola are those whose axes are parallel to the asymptotes of the hyperbola *

Apolar Theory of the Invariant Curves on Q_3

We now proceed to apply the apolar theory to the invariant curves cut out on Q_3 by the fundamental quadrics of the three systems.

(11.12) *Any two fundamental quadrics of different systems are doubly apolar. Hence all the curves Σ_1, Σ_2 are sub-rational.*

For, any quadric Q_1 is circumscribed to Δ_2, which is the common self-polar tetrahedron of all quadrics Q_2. Thus any Q_1 is out-polar to any Q_2; similarly any Q_2 is out-polar to any Q_1. That is, any Q_r is both in-polar and out-polar to any Q_s $(r, s = 1, 2, 3;\ r \neq s)$.

(11.13) *The invariant curves Σ_3 on Q_3 are apolar to the invariant curves Σ_1, Σ_2 of the other two systems.*

For, an invariant curve Σ_3 is cut out on Q_3 by a Q'_3, which we may choose to be out-polar to Q_2, without prejudice to the intersection. Supposing that Σ_1 is cut out by Q_2, it follows that Σ_1 is the contact-cyclic of Q_3 with R_2, the polar reciprocal of Q_2 with respect to Q_3. Now Δ_3 is a self-polar tetrahedron of every Q_3, in particular, of Q'_3; and since Q_2 is circumscribed to Δ_3, it follows that R_2 must be inscribed to Δ_3, and must therefore be inpolar to Q'_3 Hence, by (11.1), Σ_3 is apolar to any Σ_1, and similarly to any Σ_2.

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* It may serve as a small exercise to the reader to examine the fallacy in he following. Let S be a conic ***out-polar*** in the Cartesian sense, to the rectangular hyperbola H. Then S admits inscribed triangles ABC self-polar in regard to H. AB, AC being conjugate lines are also conjugate *circles* of H. Thus S admits triads of generating circles, namely, the tangents at A, B, C, the contact-circles of which are mutually conjugate in regard to H. Hence the *cyclics* S and H are apolar—which is in evident contradiction with the statement in the text.

The only property of Q'_3 which has been used in the proof of this theorem, is that it has Δ_3 for a self-polar tetrahedron. Hence the theorem will continue to be true if Q'_3 is not a fundamental quadric, but *any* quadric which has Δ_3 tor a self-polar tetrahedron. Now, the curves Σ_1, Σ_2 constitute two linear ∞^2-systems which do not possess any common member. Hence, they detetmine by linear combination, a linear ∞^5-system of cyclic curves; the apolar system of this ∞^5-system, must therefore be a linear ∞^2-system of cyclic curves Σ'_3. But the number of cyclic curves cut out on Q_3 by quadrics having the self-polar tetrahedron Δ_3, is also ∞^2. The curves Σ'_3 are evidently the general cyclic curves whose director circles are C_0, C_1, C_2, C_3, (the circles corresponding to the vertices of Δ_3). Hence we have:

(11 14) *The net Σ'_3 of cyclic curves apolar to all invariant curves of the first and second system, consists of all co-director cyclics of the curves Σ_3; the family Σ'_3 is also the totality of cyclic curves confocal with the curves Σ_3.*

The second part follows from the fact that confocal curves have necessarily the same director circles.

A point pair of Γ_r is an inpolar quadric of all fundamental quadrics of the rth system ($r = 1, 2, 3$). Hence by our definition of conjugate circles:

(11.15) *The isogonal (co-isogonal) circle of a given circle C may also be defined as the circle conjugate to C in respect of all invariant cyclics of the second (first) type. Ortho-circles may be defined as a pair of circles conjugate with respect to all the curves Σ'_3 that is, with respect to all invariant curves of the third type and their confocals*

The invariant hyperbolas of $A_1A_2A_3$ are special invariant curves of the third type, and their ternary polar theory agrees with their inversion-polar theory (11.2). Hence, isogonal circles being necessarily conjugate circles in respect of these hyperbolas, it follows further that isogonal

points are conjugate with regard to them, not only in the inversion-sense but also in the sense of the ternary polar theory.

The statement concerning ortho-circles in (11.1i) is redundant; for a family of confocal cyclics is a quadratic system contained in only one co-director system. Hence the ortho-circle of a given circle C is uniquely specified by the fact that it is orthogonal to the polar circle of C in respect of cyclics confocal with a particular invariant curve of the third type.

We may also derive from (11.15), the construction of (8.2) for the isogonal, co-isogonal and ortho-circles. For this, we require some lemmas on polar circles in regard to a cyclic constituted by two circles,

(11.16) The polar circle of P_1 in regard to the cyclic P^2_2 is the circle $I_{12} P_2 - \frac{1}{4} I_{22} P_1$, where the I's are the usual lineo-linear invariants in reference to the circles P_1, P_2.

For, let P_2 be $a_x b_y = a'_x b'_y$, and P_1, $c_x d_y$. The polar circle is then:

$$\begin{aligned}(a_x b_y\, a'_x b'_y,\ c_x d_y)^{1,1} &= \tfrac{1}{2}\left[(ac)(bd)\, a'_x b'_y + (ac)\,(b'd)\, a'_x b_y\right]\\ &= \tfrac{1}{2} I_{12} P_2 + \tfrac{1}{2}(b'd)\, b_y \{ (a'c)\, a_x + (aa')\, c_x \}\\ &= I_{12} P_2 + \tfrac{1}{2}(aa')\, c_x \{ \tfrac{1}{2}(b'd)\, b_y - \tfrac{1}{2}(bd)\, b'_y \}\\ &= I_{12} P_2 + \tfrac{1}{4}(aa')\, c_x\, (b'b)\, d_y\\ &= I_{12} P_2 - \tfrac{1}{4} I_{22} P_1.\end{aligned}$$

We may use this for determining the polar circle of P_1 in regard to the cyclic P_2P_3, by writing $P_2P_3 = \frac{1}{4}\{(P_2 + P_3)^2 - (P_2 - P_3)^2\}$. Using the same notation for the invariants, we immediately obtain the result:

(11.17) *The polar circle of P_1 in regard to P_2P_3 is*

$$\tfrac{1}{4}\{ 2I_{13} P_2 + 2I_{12} P_3 - I_{23} P_1 \}.$$

The special case, in which the circles P_2, P_3 are orthogonal, is important. For this case the polar circle is of the form $I_{13} P_2 + I_{12} P_3$; now $I_{13} P_2 - I_{12} P_3$ is that circle of the co-axal system (P_2, P_3), which is orthogonal to P_1, and the polar circle appears as the harmonic conjugate

of this circle in regard to P_2, P_3. Since however, P_2, P_3 are orthogonal, the harmonic conjugate is also the *inverse* with respect to P_2 or P_3. We have thus the result:

(11.18) *The polar circle of P_1 in regard to the cyclic made up of the two orthogonal circles P_2, P_3, is the inverse with respect to P_2 (or P_3) of that circle of the co-axal system $[P_2, P_3]$, which is orthogonal to P_1.*

To apply this result, we recall that the net Σ_2 of invariant curves of the second type contains six circle-pairs formed from the twelve desmic circles, and that the isogonal circle of P_1 is the circle orthogonal to the polar circles of P_1 in regard to curves of the net Σ_2. Now, the three pairs of circles (B_0B_1, B_2B_3), (B_0B_2, B_3B_1), (B_0B_3, B_1B_2) belong to Σ_2. The inverses of the polar circles of P_1 in regard to them, are clearly the isogonal lines of the joins of the vertices of the triangle with the centre of P_1; this shews that the centre of the isogonal circle is the isogonal conjugate of the centre of P_1. Taking the other three pairs of circles, namely, those on B_0B_r, B_sB_t as diameters $(r, s, t = 1, 2, 3)$, we see that the isogonal circle of P_1 is the circle orthogonal to the inverses of circles through $A_r A_s$ orthogonal to P_1, in the circles $A_r A_s B_0$ $(r, s = 1, 2, 3)$. This is substantially equivalent to (8.2).

We proceed to obtain the mutual relation between the two systems of invariant curves Σ_1, Σ_2 all of which have been shewn to be sub-rational (11.12).

(11.19) *The two systems of curves Σ_1, Σ_2 are so related that any curve of either system is apolar to the twin sub-rational curves of all curves of the other system.*

For, any two quadrics Q_2, Q_1 are also doubly apolar to one another, in addition to being doubly apolar to Q_3. Hence, Σ_1 the intersection-cyclic of Q_3 with Q_2 must be apolar to Σ'_2, the contact-cyclic of Q_3 with Q_1. Now Σ'_2 is the H-cyclic of the intersection Σ_2 of Q_1 and Q_3; hence Σ'_2

is the twin sub-rational curve of Σ_2, which proves the theorem. We shall denote the twin curves of Σ_1, Σ_2 by Σ'_1, Σ'_2 respectively.

It is clear that we may define the system Σ_1 as the totality of sub-rational curves, for which every pair of the points $B_0B_1B_2B_3$ are conjugate. Hence, by (11.8), we can define Σ_1' as the totality of sub-rational curves, for which every two of the circles through threes of the four points $B_0B_1B_2B_3$ are conjugate.

(11.20) *The linear ∞^5-system $[\Sigma_1, \Sigma_2]$ contains two and only two co-director systems π_1, π_2*

For, the ∞^5 system is cut out on Q_3 by quadrics circumscribed to Δ_3. A co-director system of cyclics is cut out on Q_3 by an ∞^3-system of quadrics with a common self-polar tetrahedron Δ'_3, Q_3 being itself one member of the system. If the system of circumscribed quadrics of Δ_3 contains such a system, it is clear that Δ'_3 must be desmic to Δ_3: from the theorem that three desmic tetrahedra are self-polar in regard to one and the same quadric, it then follows that the third tetrahedron Δ''_3 of the desmic system must also be a self-polar tetrahedron of Q_3. Hence it becomes a question of proving that any tetrahedron Δ_3, self-polar in regard to Q_3, is desmic only to two other self-polar tetrahedra of Q_3. To prove this, choose Δ_3 as the tetrahedron of reference, and let Q_3 be given by:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

Then the desmic tetrahedra whose vertices are $(a_0, \pm a_1, \pm a_2, \pm a_3)$ will be self-polar tetrahedra of Q_3, when

$$a_0^2+a_1^2 = a_2^2+a_3^2; \quad a_0^2+a_2^2 = a_3^2+a_1^2; \quad a_0^2+a_3^2 = a_1^2+a_2^2,$$

Thus there exists only two such desmic tetrahedra Δ'_3, Δ''_3, their vertices being $(1, \pm 1, \pm 1, \pm 1)$. The quadrics circumscribed to (Δ_3, Δ'_3) cut out one co-director system π_1, while the quadrics circumscribed to (Δ_3, Δ''_3), cut out the other co-director system π_2. It will be clear therefore, that the three systems of quadrics, whose common self-polar tetrahedra are respectively Δ_3 Δ'_3, Δ''_3, cut out respectively the three

co-director systems $\Sigma'_3\ \pi_1,\ \pi_2$. It is obvious that every two of these co-director systems are apolar.

The vertices of the three tetrahedra, Δ_3, Δ'_3; Δ''_3 correspond to three tetrads of mutually orthogonal circles in desmic relation. *Each circle of any one of the three tetrads thus related, must invert the two other tetrads into each other.* For, it is immediately verified that a line containing three vertices of Δ_3, Δ'_3, Δ''_3 respectively, cuts Q_3 in the Hessian pair of the triad of vertices. The statement then follows from the theorem, that any two points of a triad are separated harmonically by the remaining point and the corresponding point of the cubic-co-variant triad. Hence also it follows that the centres of the three sets of circles form three ortho-centric quadrangles, every two of which are in a cyclic quadruple perspective, the centres of perspective being the points of the remaining quadrangle. The location and properties of these quadrangles should be of great interest in the geometry of the triangle.

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Printed by Srinivasa Varadachari & Co., 4, Mount Road, Madras—21-12-'27.